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INTRODUCTION

I. OUR POINT OF VIEW

Loose talk about reform in the teaching of elementary mathematics has spread confusion and shattered the confidence of many teachers. Uneasiness and fear have developed. This mood has damaging effects in the classroom.

Teachers have been told they are teaching horse-and-buggy mathematics in a space age. New and esoteric language has begun to flood the literature and to be introduced into discussions. Some innovators have pointed the finger of scorn at traditional subject matter, pronouncing it "bad mathematics."

Relax! The mathematics our pupils have learned in traditional classrooms is good, sound, useful mathematics. It is good enough to play a central role in any moon shot. Traditional content in elementary school arithmetic is sound. It is at the core of man's advance, including his past and future bursts into space.

Our real concern is with finding solutions to two interrelated problems. How can we help pupils (a) to learn *more* mathematics, and (b) to develop a *deep appreciation* for this increasingly important part of our culture? All proposals and discussions should be evaluated in terms of their contributions to the solution of these fundamental problems.

The traditional content taught to pupils in elementary schools is completely valid. Our elementary schools must teach pupils how to carry out the fundamental operations with whole numbers and fractions and how to use these skills in solving practical problems. But the truth of the matter is that pupils simply have not mastered the traditional content. Many have too weak a grasp on facts such as that $7+2=9$ and that $8\times9=72$. (Most of these pupils will drop mathematics as soon as they can, losing contact with a tool that is essential in coping with the problems of the times.) The lack of mastery cannot be remedied just by giving more drill. Pupils will be able to master more of the traditional content only if we give them freedom to *invent* ways of finding answers, if we take steps to make sure that they *understand* what they are doing, and if they have ample opportunity to *use* what they learn.

The job of teaching more mathematics will become easier to the extent that pupils develop an appreciation for the subject. This appreciation can develop only when mathematics is learned with understanding and through active participation—it will not take place when mathematics is learned by rote. We believe that even inspired teaching of isolated fragments of mathematics will fail to produce pupils who understand and can use what they have learned.

Mathematics is a world of endless fascination. It is a world brimming with surprises, patterns, and enticing vistas. To see a pattern emerge from a conglomeration of facts, to organize seemingly disconnected ideas—these

are some of the joys of learning mathematics. It is this sense of excitement in mathematics that MATH WORKSHOP wants pupils to experience in elementary school.

To accomplish this, MATH WORKSHOP puts its *major subject matter emphasis* on *basic principles* and *unifying ideas* which tie together the facts and manipulative procedures of arithmetic. This emphasis simplifies the subject, makes it understandable, insures recall, and increases interest. Our *major pedagogical emphasis* is on teaching for discovery. We present content to pupils so that they are challenged to think about relationships, to look for patterns and clues, to draw logical conclusions for themselves, and to invent their own ways of finding answers.

II. THE SUBJECT MATTER OF MATHEMATICS

A. The Spiral Curriculum Concept in MATH WORKSHOP

The *24th Yearbook* of the *National Council of Teachers of Mathematics* talks about "the basic mathematical understandings which should be continuously developed and extended throughout the entire mathematics curriculum, grades K-12 and beyond." The concept of the spiral curriculum has been used in designing MATH WORKSHOP. At each level of the MATH WORKSHOP the main ideas of mathematics are introduced. Each successive year's activities deal with these unifying ideas with increasing sophistication.

However, pupils are not expected to master such ideas as, say, multiplication of fractions or addition of positive and negative numbers the first time they meet them. They encounter such ideas again and again, building always more depth of understanding.

The spiral curriculum rejects the "building block" approach to arithmetic that has been used in the past. Rather, the spiral curriculum calls for the early introduction of fundamental and unifying ideas with repeated reconsideration at later times. It calls for continual nurturing of important concepts year after year as understanding gradually develops.

In the early grades the hope for absolute mastery must be limited to the basic facts of arithmetic. Here, complete mastery can and must be achieved. But where big ideas are involved, mastery does not come quickly. It develops slowly and is based completely on understanding.

B. The Unifying Ideas or Strands in MATH WORKSHOP

MATH WORKSHOP emphasizes nine unifying ideas in mathematics:

1. *Structure*
2. *Sets*
3. *Number and counting*
4. *Numeration*

5. *Addition and subtraction*
6. *Multiplication and division*
7. *Functions and relations*
8. *Geometry*
9. *Measurement*

The coverage of these ideas on the pupil pages, in the Teacher's Guide, and in the practice sheets is summarized in the chart on pages 48-49. Each Teacher's Guide page lists the unifying ideas considered on the corresponding pupil page and contains many suggestions for extending these ideas beyond what is called for on the pupil page. Let us consider each of these unifying ideas.

1. Structure

Suppose that you came upon some sentences from a Martian arithmetic book:

1) | ☆ □ = □ ☆ |
2) ⊓ ☆ ◪ = ◪ ☆ ⊓
3) (⊠ ☆ ⌜) ☆ □ = ⊠ ☆ (⌜ ☆ □)
4) (□ ☆ ◪) ☆ | = □ ☆ (◪ ☆ |)
5) □ ☆ ⊠ = □
6) ◪ ☆ ⊠ = ◪

Somehow you have found out that the symbols:

|, □, ⊓, ◪, ⊠, ⌜

represent different cardinal numbers. You guess that the symbol ☆ is an operation sign, and you wonder which operation it stands for. Sentences 1 and 2 convince you that ☆ stands for neither subtraction nor division, for these sentences suggest that ☆ stands for an operation for which order does not affect the results. Sentences 3 and 4 also support this conviction because they suggest that grouping does not affect the results. However, there is nothing about sentences 1-4 which would help you choose between addition and multiplication as the operation denoted by ☆. Perhaps the other sentences will help.

Sentences 5 and 6 turn out to be of little help. In fact, both of them suggest that ☆ could denote addition, multiplication, subtraction, or division. Fortunately, sentences 1-4 rule out both subtraction and division. If ☆ stands for addition, then ⊠ denotes the number 0; if ☆ stands for multiplication, then ⊠ denotes the number 1.

It turns out that, based on what can be learned from sentences 1-6, the operation sign ☆ could stand for addition or it could stand for multiplication.

We can't decide from sentences 1-6 whether it is addition or multiplication that is denoted by ☆ because these operations have certain properties in common and these properties are reflected in sentences 1-6. The *additive*

structure of the number system is similar to its *multiplicative structure.* Both operations are commutative (sentences 1 and 2) and both are associative (sentences 3 and 4). Also, each operation has an identity element (sentences 5 and 6).

Whenever you provide exercises for pupils that bring any of these properties to the surface, you are paying attention to the structural properties of the number system. For example, exercises such as the ones which ask pupils to fill the frames to make the following into true sentences:

$$8 \times 7 = \bigcirc \times 8$$
$$7 + (3 + 5) = (7 + \triangle) + 5$$

are exercises which help pupils develop an awareness of commutativity and associativity. Another such exercise is one which asks pupils to complete the addition *Cross Number Puzzle:*

4	3	
5	1	

Notice that the cell in the lower right-hand corner gets filled with the same number whether the pupil adds the numbers in the rows and then adds the sums:

4	3	7
5	1	6
		13

or adds the numbers in the columns and then adds the sums:

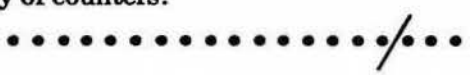
4	3	
5	1	
9	4	13

This invariance of the total highlights the rearrangement properties of addition.

Of course, the advice that one should give pupils exercises which bring structural properties to the surface does not necessarily mean that pupils need to be told the names for these properties. A pupil who has discovered these properties by working with displays of physical objects has a sound understanding of the properties. He thinks of them almost as laws of nature. A young pupil who tells you that $7 + 3$ is the same number as $3 + 7$ *because* "addition is commutative" has surely been coerced into using these words. He knows that $7 + 3 = 3 + 7$ because any display of objects which represents $7 + 3$ can be rearranged (without changing the total number of objects) into a display which represents $3 + 7$.

A pupil can be aware of and can use the structural properties without knowing their names. Exercises can be found which encourage pupils to make use of the properties. For example, a number line exercise which asks pupils to find the landing point when the starting point is 3 and the jump is 16 to the right could be solved by starting at 16 and jumping 3 to the right. The fact that this alternative procedure gives the correct answer strengthens a pupil's awareness of the commutative property of addition and helps to convince him of its usefulness.

If a pupil finds the sum of 16 and 3 by finding the sum of 6 and 3 and then finding the sum of 10 and 9, he has made use of the associative property of addition in a helpful way. (Of course, he has also made use of the fact that the numbers are expressed in decimal notation.) He does not become convinced of the correctness of his technique by being told that it is based on the fact that addition is an associative operation. Rather he becomes convinced by observing that this display of counters:

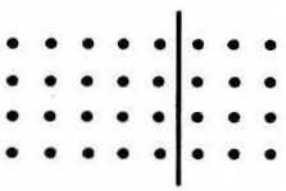


can be rearranged into this display:



In addition to the structural properties of commutativity, associativity, and the existence of an identity element for addition, the operation of adding a number has the property that it can be "undone" by subtracting the same number. Similarly, the operation of multiplying by a (nonzero) number can be "undone" by dividing by that number.

Pupils also learn that certain multiplication facts can be discovered by using other multiplication facts and addition. For example, pupils can find the product of 8 by 4 by adding the products of 5 by 4 and 3 by 4. That this procedure is valid is based on the "physical" fact that an 8 by 4 array of counters can be partitioned into a 5 by 4 array and a 3 by 4 array:



This experience helps pupils become aware of the fact that multiplication distributes over addition—a property of multiplication and addition. The term "distributive property" is not used on the pupil pages and need not be used in class. The important thing is that pupils use this property in deriving certain products from others.

2. Sets

The principal contribution of sets to elementary arithmetic is that they can be used to clarify the notion of cardinal number. For a long time

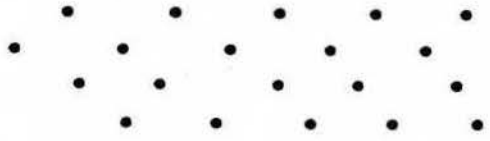
people thought that the word "two" in the expression "two apples" played the same kind of role as the word "green" in the expression "green apples." This was the cause of much confusion about the nature of cardinal numbers. An individual apple can have the property of "green-ness" but it can't have the property of "two-ness." Only sets can have the property of number. A set consisting of two green apples is a set whose cardinal number is two. Also, any set whose members can be matched with those of a set of two apples is a set whose cardinal number is two. From here it is an easy step to say that the cardinal number two is the class of all such sets.

Now, a set is an abstract entity just as a number is abstract. You can see two apples but you cannot see a set of two apples any more than you can see the number two or justice or liberty. For example, it is complete nonsense to ask the *set* of boys in the classroom to stand or to ask pupils to look at the *set* of counters on the table. There will come a time in the pupil's mathematical education when he will need to use the word "set" but we do not think it is a good idea for him to develop in elementary school the notion that a set is a tangible object. Therefore, we have rarely used the word *set* in the Pupil Book, and have used it sparingly in the Teacher's Guide. Instead, we have used nontechnical terms such as *display*, *group*, and *array*.

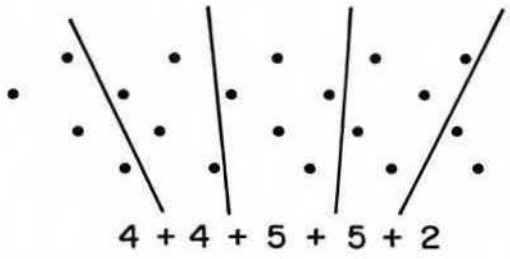
Teachers who wish to read a discussion of set language and notation as used in mathematics can refer to the Appendix of this book as a supplementary unit of study.

3. Number and Counting

In the primary grades the basic mathematical idea is that of counting. The operations of addition and multiplication are presented as shortcuts to counting. Given a display of objects:



and asked to tell how many objects are in the display, a pupil might proceed to count them one by one. This is the most primitive approach. A more sophisticated approach is to partition the display, obtain the count for each of the smaller displays, and add the counts:



An even more sophisticated approach is to rearrange the objects in the display into a rectangular array:



If one is familiar with the count for a 4 by 5 array, he knows immediately that the original display contained 20 objects.

The study of fractions and negative numbers is considered an extension of the study of number and counting in this organization of unifying ideas.

A word about our treatment of fractions in Level E is in order. We present fractions as pairs of multiplication and division operators. Thus, to find ⅔ of 6 feet, one can either first multiply 6 feet by 2 and then divide the result by 3 or first divide 6 feet by 3 and then multiply the result by 2. This approach to fractions as pairs of operators is not only mathematically sound but is very much in consonance with the way fractions are used in everyday life.

4. Numeration

The idea of numeration grows out of the need to report a count, and the mode of reporting is culturally determined. For example, if the objects in the display shown above represent glasses of milk, we might report "1 gallon and 1 quart" in answer to the question: How many? If the objects represent eggs, we might report "1 dozen and 8 more" or "4 less than 2 dozen."

Each of these reports is a description of the result of some partitioning procedure. When decimal partitioning is used, the English language provides us with words for giving brief reports. For example, the report "3 ten-groups and 7 one-groups" or "3 tens and 7 ones" can be abbreviated to "thirty-seven" and can be recorded in positional notation by the symbol "37."

Much activity in the study of denominate (measurement) numbers is provided at all levels of MATH WORKSHOP to help pupils understand that the results of a count may be recorded in other bases than base ten. They come to realize that systems of numeration are arbitrary, but that each has a pattern which can be used to keep matters straight.

5. Addition and Subtraction

The operation of addition is developed as a shortcut to counting. Given enough opportunities to find the sum of 5 and 4 by counting, for example, a pupil soon commits to memory the fact that $5+4$ is 9. He makes use of

this fact as well as of positional notation and associativity when finding the sum of 25 and 4:

$$25+4=(20+5)+4=20+(5+4)=20+9=29$$

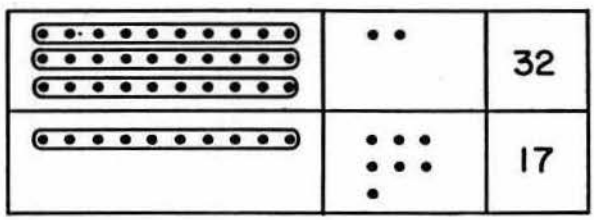
The addition facts from "$0+0=0$" through "$9+9=18$" should be memorized during the primary grades.

We use the word *subtraction* to denote the process used to undo the result of adding. So, for example, to compute 8–3 amounts to finding the number whose sum with 3 is 8. From this point of view, it is natural to learn subtraction facts at the same time one is learning addition facts. Thus, we have *fact teams*, groups of four sentences, such as:

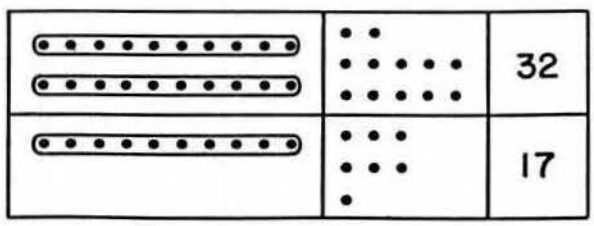
$$\begin{array}{ll} 5+3=8 & 8-3=5 \\ 3+5=8 & 8-5=3 \end{array}$$

which are derived from one addition fact with the help of commutativity and the fact that subtracting a number is the *inverse* of adding that number.

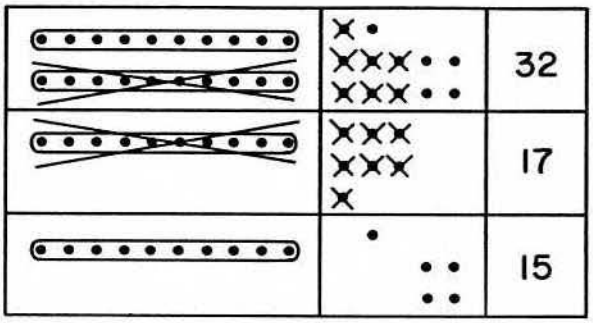
Addition and subtraction algorithms are developed as the end products of activities in which pupils manipulate physical objects and keep records of the manipulations. An algorithm becomes a record-keeping device for imagined manipulations. Thus, from the viewpoint of the pupil, an algorithm works because of laws which govern the manipulation of physical objects. For example, consider an algorithm used in subtracting 17 from 32. In terms of physical objects, one can find the difference by counting what remains when 17 objects are removed from a display of 32 objects. We start with decimally partitioned displays of 32 objects and 17 objects:



In order to carry out the take-away step, we need to break into one of the ten-groups. So, we rearrange the display for 32 by exchanging 1 ten-group for 10 one-groups:



and then take away 1 ten-group and 7 one-groups:

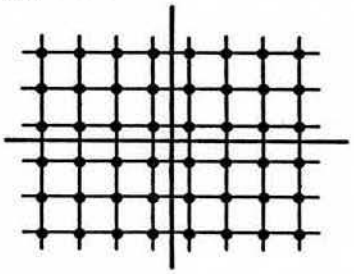


These three stages are recorded in the steps of the algorithm:

$$\begin{array}{r} 32 \\ -17 \\ \hline \end{array} \qquad \begin{array}{r} 2\not{3}12 \\ -17 \\ \hline \end{array} \qquad \begin{array}{r} 2\not{3}12 \\ -17 \\ \hline 15 \end{array}$$

6. Multiplication and Division

Like addition, multiplication is developed as a shortcut to counting. In this case the counting involves objects arranged in rectangular arrays or otherwise grouped to encourage counting by groups. For example, counting the spaces in an egg carton and making 5 number line jumps of 2 to the right, starting at 0, are multiplicative situations. By continual practice in the primary years a pupil should commit to memory the multiplication facts from "$0\times0=0$" to "$9\times9=81$," and each of these facts should be discovered by the pupil. Some of the facts should be discovered by direct counting, but the discovery of others should involve partitioning of arrays and adding the counts of the parts. For example, a pupil might learn that 3×4 is 12 by counting the spaces in a certain type of egg carton. He can use this fact to learn that 6×8 is 48 by drawing 6 horizontal lines and 8 vertical lines and then partitioning the resulting display of crossing points:



$$\begin{array}{r} 12 \\ 12 \\ 12 \\ 12 \\ \hline 48 \end{array}$$

Division is the process used to undo multiplication. So, for example, to compute $24\div8$ amounts to finding the number whose product with 8 is 24. From this point of view, it is natural to learn division facts at the same time one is learning multiplication facts. Thus, as in the case of addition and subtraction, we have facts teams for multiplication and division such as:

$$\begin{array}{ll} 8\times3=24 & 24\div3=8 \\ 3\times8=24 & 24\div8=3 \end{array}$$

which are derived from one multiplication fact with the help of commutativity and the fact that dividing by a (nonzero) number is the *inverse* of multiplying by that number.

7. Functions and Relations

An example of a function is the operation addition-of-2. This function assigns a unique number to each number. For example, it assigns 5 to 3, 19 to 17, and 202 to 200. A common way to exhibit a function is to show a table which lists some of the pairings. For example, to exhibit the function addition-of-2, we might show the table:

3	4	9	12	18	30
5	6	11	14	20	32

The numbers listed in the top row are the numbers on which the function operates. Those listed in the bottom row are the numbers we obtain from applying the function. The use of two colors helps to distinguish "input" from "output." By reversing input and output, this same table shows another function—subtraction-of-2—which is the inverse of addition-of-2.

All functions are examples of mathematical ideas known as relations. Numerical relations which are not functions are those which assign more than one number to some given number. For example, the greater-than relation assigns 4, 5, 6, etc., to the number 3.

8. Geometry

Under this general heading, one finds in MATH WORKSHOP a variety of topics—classification of common figures, comparisons of area and perimeter, partitioning of regions, similarity transformations (the doubling and halving of line segments), the study of triangulation, angles, parallel lines, and circles.

9. Measurement

Under this heading is found work with denominate numbers and the conversion of units. Further activity is provided in measurement of geometric shapes.

C. Another View of Mathematics: Counting and Shortcuts to Counting

You probably have noticed that a basic and simple notion recurs as a central theme in the previous discussion of the nine unifying ideas. It is that, to a large extent, elementary school mathematics is essentially *counting and shortcuts to counting.*

The authors of MATH WORKSHOP hope that all pupils will see arithmetic as a very simple subject. Pupils will develop basic confidence in their ability to move ahead if they are helped to realize that they have but a few easy tasks:

1. Learn how to COUNT
2. Learn how to RECORD RESULTS of counting
3. SOLVE PROBLEMS involving counting
4. Find STRATEGIES to avoid one-by-one counting

That's all there is to it!

Every pupil in elementary school ought to adopt the slogan: When in doubt, COUNT. He ought not look to the teacher, or the text, or his friends for assurance about his statements in arithmetic. Rather, he ought to look at the world of things . . . What happens there?

He ought to learn that there are many reliable ways to get to the solution to any exercise or problem—that he ought to use the way that seems most reasonable. You can help him understand this important freedom to find his own way and to use it. We try to give the pupil the equipment he needs to solve any problem so that, given enough time, he can, if all else fails, count his way to the answer.

Thus, our central purpose has been to give every pupil, from the fastest to the slowest, the confidence that he can always get right answers in computations. He must learn the strategies that are available to make the job easier, but he must understand them. It is the central purpose of MATH WORKSHOP to help pupils develop the confidence and ability to perform the four tasks enumerated above.

Lack of confidence is usually the real stumbling block to those who are called "slow learners." They are convinced they can't remember or they are afraid they will forget, so they try to avoid doing arithmetic. It never occurs to many of them that they can *always be right* if they count, and that they need use only those shortcuts or strategies they understand.

Slow learners will be given a new lease on their mathematical lives as we help them realize that there is less to memorize in this subject than any other. It all begins with counting and with thinking about what there is to be counted. Beyond that, if they learn the 390 basic facts of addition, subtraction, multiplication, and division, they can develop really effective strategies to avoid the hard work of one-by-one counting.

Fast learners need to learn about the other side of the coin. Since there is little that needs memorization and lots to think about, it is impossible to memorize your way through the subject. At some point along the way, the mind will rebel at all the memorization required—and the person, regardless of his ability to memorize, will become a "mathematical drop-out." The fast learner needs to realize that, since the subject is basically so simple, he can make important side-trips on his own, unafraid of pitfalls, confident he won't get lost.

III. HOW TO TEACH MATHEMATICS

A. Adopt a Sound Learning Theory

Every conscientious teacher has a theory of how pupils learn arithmetic. It is the sum of all the reasons behind his behavior in the classroom. If he asks the pupils to open the book to page 42 and do the even examples, he feels that activity will promote learning. If he tells Billy to write "$4 \times 7 = 28$" twenty-five times, he is convinced that the repetition will help Billy remember that fact the next time he is asked for the product of 4 and 7. If, instead, he asks Billy to count the number of days in 4 weeks, he feels that relating multiplication facts to recognizable events in the world he lives in will help Billy remember that $4 \times 7 = 28$.

If a teacher could get outside himself and analyze his teaching of just one mathematics class, he could then write down the theory of learning that guides him.

The authors of MATH WORKSHOP have a viewpoint on learning in mathematics. They recognize three different levels of learning experiences:

1. Rote Learning
2. Problem Solving
3. Independent Investigation

They believe that most failures and difficulties in arithmetic are the result of a conscientious, but misguided, emphasis by teachers and textbooks on rote memorization. The authors believe that the central task before us is to present arithmetic as a problem-solving activity and with unlimited opportunities for independent investigation.

1. Rote Learning

We share the ability to learn by rote memory with all animals. Pavlov's dogs learned by endless repetitions. Horses have been trained to point a hoof at 5 when the trainer calls out "two plus three." A psychologist, half in fun, pointed out that a stick also learns in this way—put a notch in a stick today and it will be there tomorrow.

Pages of random repetitions hope to educate students on the level of rote memory. If one page doesn't do the job, then let's try two. If pupils persist in forgetting, drill some more!

Actually, when pupils rebel openly or in any of the less obvious ways they are adept at inventing, that rebellion is usually a sign of an active mind. The human mind, after all, has a sense of dignity that is offended by long stretches of drill bent on rote memory as the objective. We should not be surprised when pupils turn away from assignments that recognize only the lowest level on which the human mind can operate.

Efforts to teach by rote memory have always been discouraging. Pupils don't respond "properly," so they are given larger doses of the same. In

desperation, the driller sooner or later reaches the conclusion that the pupil can't learn. The situation is further darkened by the mistaken theory that slow learners can only learn on the level of rote memory.

2. Problem Solving

The human mind is unique in its ability to solve problems—to face a question and announce: I don't know, but I can find out!

First graders are protected from questions such as "19+3=____" because the rules one needs to solve this problem by rote memory are not yet available. However, if a first grader can count as far as 22 and if he understands what is meant by addition, while he doesn't know the answer, he can find it. A consequence of teaching arithmetic at the level of rote memory in first grade was revealed when one of the authors asked a large group of beginning second graders to answer this question, asked in both forms they might have encountered:

$$\begin{array}{r} 19 \\ +\ 3 \\ \hline \end{array} \quad \text{and} \quad 19+3=\_\_\_\_$$

None were familiar with the horizontal form. All wrote the same result . . . without exception:

$$\begin{array}{r} 19 \\ +\ 3 \\ \hline 112 \end{array}$$

They read this statement as: Nineteen plus three equals one hundred twelve. They were asked: If you had nineteen cookies and I gave you three more, would you have one hundred twelve cookies? Most answered: No! I would have twenty-two cookies.

These second graders had been deprived of the opportunity to solve problems and were already victims of their most unreliable faculty—the ability to learn by rote.

First graders can understand and solve problems designed to give them opportunities to use the simple facts of addition. Here is a question that almost every adult has to think about:

> How many ways do you know to complete the following as a true statement?
>
> $$\_\_\_\_+\_\_\_\_=3$$

(We assume the pupil's acquaintance with numbers is limited to zero and the counting numbers.)

The pupil does not know the answer to the question . . . but he can find out—unless he has been trained to limit his response to "I don't know" when

he doesn't remember. A pupil who has often been encouraged to find out when he can't remember will make a list. He may write:

$$2+1=3$$
$$0+3=3$$
$$1+2=3$$
$$3+0=3$$

Is that all there are?

In response to this question, one first grade pupil simply reorganized the list without saying a word and took her seat.

$$3+0=3$$
$$2+1=3$$
$$1+2=3$$
$$0+3=3$$

This pattern was certainly an eloquent argument that there are no more!

Here is another first grade question that almost no adult can answer from memory. Yet it is a question that every adult and more first graders can work out if they make the effort:

> How many ways are there to complete the following as a true statement?
>
> $\_\_\_\_+\_\_\_\_+\_\_\_\_=3$

(Again assuming the problem solver is limited to zero and the counting numbers.)

Notice that in "finding out" one uses or practices basic combinations in addition. Other similar problems could require use of or practice in all basic combinations. All the random exercises in memory that constitute "drill" have now been encountered incidentally along the way toward the solution of problems.

> How many numbers can be expressed by one of the following or as the sum of two or more of them—none to be used more than once in any sum?
>
> 4, 2, 16, 1, 8

Only those who are already familiar with this list have any idea of the surprises that lie ahead. We begin a list, expressing each number as one from this list or as the sum of some combination of numbers from the list:

$1=1$	$6=4+2$	$11=\_\_\_\_\_$
$2=2$	$7=4+2+1$	$12=\_\_\_\_\_$
$3=2+1$	$8=8$	$13=\_\_\_\_\_$
$4=4$	$9=8+1$	$14=\_\_\_\_\_$
$5=4+1$	$10=\_\_\_\_\_$	$15=\_\_\_\_\_$

As we extend the list, we find no gaps until we come to 32. But that is one more than the sum of the entire list.

Now, we go back to see how many can be expressed in more than one way (not counting changes in order). We write:

$7=4+2+1$ ("$7=2+4+1$" is considered the same combination)

Is there another combination from our list we might have used? One way to indicate 22 is "$16+4+2$." Is there another combination whose sum is 22?

We try many combinations—using or practicing addition—and eventually develop a suspicion that each can be expressed in only one way as one of the numbers listed or as a sum of the numbers listed.

The authors of MATH WORKSHOP know that there is no end to such interesting problems in mathematics—problems that can be solved by pupils who are encouraged to say: I don't know, but I can find out.

We have assumed the responsibility of building a Pupil Book and designing practice sheets that will help teachers and pupils approach the subject of mathematics as being basically concerned with solving problems. When the pupil recognizes his ability to solve problems, he responds with enthusiasm, and his power to recall basic facts and processes he has encountered flourishes.

3. Independent Investigation

Once the pupil has developed a confidence that he can find things out for himself, he may be so delighted with his accomplishments that he will look around for problems to solve, and may invent problems of his own. He will not be satisfied with the list 4, 2, 16, 1, 8, which he used to write all the numbers 1 through 31. He will want to expand the list so he can go beyond 31. What number or numbers should he add to the list? Or, he might like to try an entirely different list such as 1, 4, 7, 9, 18, which behaves similarly but with important differences.

When pupils begin posing problems for themselves and solving them, they will have a taste of intellectual excitement that will lead them on and on. The teacher who has helped a pupil to use his mind on this level deserves the thrill that comes to him.

The authors of MATH WORKSHOP are dedicated to helping elementary school teachers rescue the subject of mathematics from emphasis on rote memorization and present it as basically a problem solving activity. Further, we have tried to present many opportunities that will encourage some pupils to pose questions to themselves and to solve them.

B. Pose Challenging Questions

1. "What Can You Say?"

The Report of the Cambridge Conference on School Mathematics said:

> "We hope that many problems can be found (we know a few) that read, 'Here is a situation—think about it—what can you say?' "

The authors of MATH WORKSHOP share that hope and have built the pupil pages, the accompanying practice sheets, and notes for teachers around such problems. A mathematical problem is a situation worth thinking and talking about, a situation that suggests exploration and investigation. Consider this situation:

What Can You Say? Situation 1

> Make a list of the square numbers up to 100 . . .
>
> $0\times0,\ 1\times1,\ 2\times2,\ 3\times3,\ \ldots,\ 10\times10$
> 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

Making the list is simply computation . . . it is not a problem. But this list provides situations we can think and talk about. For the numbers in this list:

Is there a pattern of even and odd numbers? (Even and odd numbers alternate.)

By which square number (other than 1) can you divide evenly each even square number? (4)

What happens if we divide the odd square numbers by 4? (All odd square numbers give a remainder of 1 when divided by 4.)

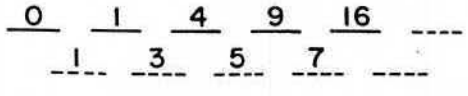
Can we find a pair of numbers in the list whose sum is also in the list? (Yes. Two such pairs are: $9+16=25$ and $36+64=100$.)

Are there any more? Could we find any more if the list were extended to include 121, 144, 169, etc.?

Can we find a number in the list that is twice as large as another in the list? Three times as large as another? (After a lot of work, providing meaningful practice, we find that the answer to both questions is "no!")

Is there a number in the list that is four times as large as another in the list? (Surprise!)

What can be observed about the differences between successive terms?



Will this pattern continue?

Each of the questions asked about this situation is simple and direct. If this is a pupil's first encounter with this situation, the answers have not been memorized. Yet each answer is well within the reach of any pupil who could have

created the list in the first place. He need only be ready to find out for himself by putting to work what he already knows.

How does one invent situations worth this kind of consideration? Unexpected sources for such situations are old arithmetic text books and current standardized tests . . . with a *little* modification. Both sources are, for different reasons, limited in their concern to simple, direct answers, so their questions do not lead to *situations worth considering.* They both suffer from a need or compulsion to give too much information. Here is an example: "Adult tickets are 85¢ each and children's are 45¢ each. *How much will 2 children's and 3 adult's tickets cost?*"

As is, there is nothing to think about . . . it only requires a bit of computation. The answer is $3.45 . . . and we go on to the next example.

This might be called the "customer" approach to such situations—one purchase, and that's the end of it. The clerk, or box office ticket seller, will have a different view of the matter. Here is his situation:

What Can You Say? Situation 2

Adult's tickets—85¢ each Children's tickets—45¢ each

He might make a pair of charts to help him solve the many problems he will face as different people make different purchases:

No. adult's tickets	1	2	3	4	5	6
Price	85¢	$1.70	$2.55	$3.40	$4.25	$5.10

No. children's tickets	1	2	3	4	5	6
Price	45¢	90¢	$1.35	$1.80	$2.25	$2.70

If his charts are long enough, he is now ready to compute each sale by looking up not more than two entries and adding them, perhaps in his head. If a customer wanted 3 adult and 2 children tickets, he would look at his charts

Adult's	3
	$2.55

Children's	2
	90¢

and ask the customer for $3.45. He might, however, make a chart to avoid even his addition. It would have to be quite a large chart. One corner of it would look like this:

Children 45¢ each

Adults 85¢ each	0	1	2	3	4	5
0	~~0~~	45¢	90¢	$1.35	$1.80	$2.25
1	85¢	$1.30	$1.75	$2.20	$2.65	$3.10
2	$1.70	$2.15	$2.60	$3.05	$3.50	$3.95
3	$2.55	$3.00	$3.45	$3.90	$4.35	$4.80
4	$3.40	$3.85	$4.30	$4.75	$5.20	$5.65
5						

One of his other charts is now in the top row of amounts. The other is in the left hand column. Other entries show the sums of the amounts at the top of the column and the left end of the row. If this chart were extended far enough, the ticket seller would have only to know the number of each kind of ticket to be purchased to find the entry giving the total amount to be charged for any purchase.

Notice that all entries in the chart above are different. So, if the amount of the purchase is known, one can find out exactly how many of each kind of ticket was purchased. If a purchase was \$2.20 and the chart was not available, it might require several trials to find that it could have been 3 children's and 1 adult's tickets and probably more trials to be sure that no other combination would fit.

We might ask if it is possible for a purchase to be exactly \$5.00. Without the chart, this would lead to a lot of computation before we found that the closest we can come to spending exactly \$5.00 is to spend \$4.95 for 11 children's tickets.

What is the smallest amount that can be spent in more than one way? (\$7.65 . . . 17 @ 45¢ or 9 @ 85¢.)

Remember that this investigation started from part of a typical "story problem." It was converted from a customer problem to an investigation of the box office salesman's point of view.

Here are several additional *What Can You Say?* situations, some taken out of mathematics itself, others taken from applications of mathematics to the real world. You can use each as a model for creating many others like it. Each suggests many similar situations that would bear independent investigation.

What Can You Say? Situation 3

Here are three paper cups:



Helen had four pieces of chalk. She put them in the cups.

Here is a situation. Think about it. What can you say? This and somewhat more pointed questions will elicit such observations as:

If two cups are empty, the other has 4 pieces in it.

There cannot be the same number of pieces of chalk in each cup.

There might be a different number of pieces of chalk in each cup—3 in one, 1 in another, and 0 in the third.

At least one cup has 2 or more pieces of chalk in it.

What can you say if Helen put only one piece of chalk in the cups?

Two cups are empty.

There cannot be the same number of pieces of chalk in each cup.

What can you say if she put 2 pieces in the cups?

There cannot be the same number of pieces of chalk in each cup.

They cannot all be different.

Two cups must have the same number of pieces in them—either two cups are empty or two cups have exactly one piece of chalk in each.

What can you say if Helen puts 3 pieces in the cups?

They can all contain the same number of pieces of chalk.

Each can have a different number of pieces of chalk —one empty, one with a single piece, and one with two pieces.

If two cups have the same number and the third has a different number, then two cups are empty.

What can you say if Helen puts 7 pieces of chalk in the cups?

They can't all contain the same number.

They can all have different numbers.

In at least one cup, there are at least 3 pieces of chalk.

If no cup has as many as 4 pieces in it, then one has 3 pieces and the others have 2 in each.

It could be that each cup that is to the right of another has twice as many pieces as the cup to its left. (1, 2, 4)

What Can You Say? Situation 4

Bill asked his father if he had change for a quarter. "Yes. What combination of coins do you want?"

Here is a situation. What can you say?

> How many of each kind of coin would Bill's father need in order to make change in any way Bill might ask? We might make a list of all possible combinations, following some plan that will help us know when we are done:

Dimes	2	2	1	1	1	1	0	0	0	0	0	0
Nickels	1	0	3	2	1	0	5	4	3	2	1	0
Pennies	0	5	0	5	10	15	0	5	10	15	20	25
Coins	(3	7	4	8	12	16	5	9	13	17	21	25)

> Bill's father would need 32 coins: 2 dimes, 5 nickels, and 25 pennies; with these 32 coins, he could make change in any of the above ways.
>
> There are 12 different ways to make change for a quarter.

(Do you agree with the authors that many of these explorations would make excellent independent activities, homework activities, or activities for exploration by teams of pupils?)

If we make a record (like the one above) of the number of coins in each combination, we notice that:

> Since no two combinations involve the same number of coins, we only have to know how many coins were used in order to tell which combination was used.
>
> There is no way to make change for 25¢ using combinations with exactly 2, 6, 10, 11, 14, 15, 18, 19, 20, 22, 23, or 24 coins.

How does the problem change if we investigate ways to make change for a half-dollar?

1. We already know that there would be 12 combinations using just one quarter (one quarter plus each of the 12 combinations listed above) and we know that there is only 1 combination with 2 quarters . . . a total of 13 thus far.
2. Without using quarters, we might list combinations using a specific number of dimes:

   5 dimes . . . . 1 combination
   4 dimes . . . . 3 combinations
   3 dimes . . . . 5 combinations
   2 dimes . . . . 7 combinations
   1 dime . . . . 9 combinations
   0 dimes . . . 11 combinations
   36

3. This gives 1+12+36, or 49, combinations.
4. All amounts 30¢ and more can be made by at least 2 combinations that have the same number of coins.

30¢ . . . 5 pennies and 1 quarter (6 coins)
6 nickels (6 coins)
40¢ . . . 1 quarter and 3 nickels (4 coins)
4 dimes (4 coins)
1 quarter and 15 pennies (16 coins)
6 nickels and 10 pennies (16 coins)

Perhaps someone would like to investigate the ways to make change for $1.00 or $5.00. Or someone might notice that 90¢ can be made with at least 4 different combinations of 18 coins.

What Can You Say? Situation 5

SALE: Pencils . . . 5¢ each
Pens . . . 9¢ each
Notebooks . . . 25¢ each

Limit: 4 of each to a customer

What can you say about purchases that can be made under the terms of this sale?

What would be the largest purchase? (4 of each. Since 1 of each would be 39¢, 4 of each must be $1.56.)

Could a 30¢ purchase be made? (Yes. One way would be to buy a pencil and a notebook.) In what other way? (None)

How might the amounts indicated below have been spent? The record below shows one combination for each amount.

Amount of Purchase	23¢	53¢	76¢	59¢	81¢	89¢	$1.37
No. of Pencils	1	2	3	0	4	1	2
No. of Pens	2	2	4	1	4	1	3
No. of Notebooks	0	1	1	2	1	3	4

What other combinations of items, other than those indicated above, give purchases of 23¢, 53¢, 76¢, 59¢, 81¢, 89¢, or $1.37? (None)

Is it possible to spend $1.25 under the terms of this sale? (No) How many amounts of money can be spent in two or more different ways under the terms of this sale? (Surprisingly enough, every one of the

124 different combinations of items possible under the limit set has a different total price. Thus, if the amount of the purchase is known, the exact combination of items can be found.)

What amounts of money more than 10¢ and less than a dollar cannot be spent under the terms of this sale? (11¢, 12¢, 16¢, 17¢, 21¢, 22¢, 26¢, and 31¢. All other amounts up to $1.25 can be spent.)

Below is a summary of purchases of all combinations of pencils and pens, with no notebooks and with a single notebook:

No Notebooks

Pencils at 5¢

Pens at 9¢	0	1	2	3	4
0	~~0~~	5¢	10¢	15¢	20¢
1	9¢	14¢	19¢	24¢	29¢
2	18¢	23¢	28¢	33¢	(38¢)
3	27¢	32¢	(37¢)	42¢	47¢
4	(36¢)	41¢	46¢	51¢	56¢

One Notebook and

Pencils at 5¢

Pens at 9¢	0	1	2	3	4
0	25¢	30¢	35¢	(40¢)	45¢
1	34¢	(39¢)	44¢	49¢	54¢
2	43¢	48¢	53¢	58¢	63¢
3	52¢	57¢	62¢	67¢	72¢
4	61¢	66¢	71¢	76¢	81¢

Additional charts can be made, noting the combinations including 2 notebooks, 3 notebooks, and 4 notebooks. No amount will occur twice—all combinations of items have a unique total price; thus, if we know the amount spent and can find it in the chart, we can easily determine the items purchased.

Notice the five entries circled in the charts above: 36¢, 37¢, 38¢, 39¢, and 40¢—an over-two-and-up-one pattern (the Knight's move in chess). Also notice that 41¢, 42¢, 43¢, 44¢, and 45¢ follow the same over-two-and-up-one pattern. This may help suggest that no amount will be repeated.

If the prices are changed to

Pencils	3¢
Pens	4¢
Notebooks	16¢

and a limit set of 3 each to a customer, the items purchased by a given price are unique, as in the situation discussed above. The smallest purchase is 3¢. The largest purchase is 69¢. All amounts of money between 3¢ and 69¢ can be spent in one and exactly one way except 5¢, 64¢, 67¢, and 68¢—which cannot be spent for any combination of items.

What Can You Say? Situation 6

Suppose each of 6 pupils was asked to contribute his "favorite" number to a list and these numbers were recorded:

15, 4, 26, 1, 44, 9

What can you say about that list?

Three of them are odd and three are even.

None is prime.

Their sum is 99.

There is no combination of these numbers whose sum is 32, 33, or 34 (assuming that no number is included more than once in each combination).

$27 = 26 + 1$	$32 =$ _____
$28 = 15 + 9 + 4$	$33 =$ _____
$29 = 15 + 9 + 4 + 1$	$34 =$ _____
$30 = 26 + 4$	$35 = 26 + 9$
$31 = 26 + 4 + 1$	$36 = 26 + 9 + 1$

How many different numbers can be expressed as the sum of a combination of two or more of those listed? (57)

Can any member of the list be expressed as the sum of some combination of others from the list? (The answer is no. We can come close: $15 + 26 + 4 = 45$, but not 44; $1 + 9 + 15 = 25$, but not 26.)

Can any number be expressed as the sum of a combination of two or more numbers from the list in more than one way? (No. The sum of every combination is different from the sum of every other combination.)

Can you find another list of six numbers such that the answers to the last two questions are No? (The simplest is 1, 2, 4, 8, 16, 32.) There are an unlimited number of such lists but they are hard to find unless the numbers are separated by large gaps, for example: 1, 17, 101, 900, 2000, 7587. In fact, any list in which each new number is more than the sum of all preceding numbers will work. Here are a few that are not at all obvious:

4, 6, 7, 8, 20, 40
1, 4, 7, 9, 18, 36
1, 4, 6, 8, 16, 32

A newcomer to this problem will hurry through hundreds of examples of addition without noticing the arithmetic practice. His goal is the solution of a problem—the practice is incidental.

It's easy to ask a question so easy that almost anyone can answer it. And it's easy to dig up trick questions—there are books full of them. But challenging questions, like those in the *What Can You Say?* situations just presented, are those that encourage independent research and lead to discovery. They must be neither too hard nor too easy, and must not rely on

clever tricks. Rather, challenging questions are less than obvious but will yield answers if the pupil thoughtfully applies what he already knows.

Such questions are hard to come by. They are worth their weight in gold. Often they are hidden in very simple situations. In this section and throughout the MATH WORKSHOP materials, we have presented the largest collection of such problems that have ever appeared in elementary school textbooks.

2. "What's My Rule?"

In first grade, Miss Atkins tells the pupils that she has a secret rule (a function) and that she will help them read her mind. "Call out a number and I will answer with a number."

"Five," a pupil calls out.

"Four," Miss Atkins answers. "Someone call out another number."

"One," a pupil calls out.

"Zero," the teacher replies.

"Seven," another pupil calls out.

"Six," the teacher answers. "Have you guessed my rule?"

"Yes," Bill is quick to reply.

"Don't give my secret away. Just call out a number and tell me what you think I will reply."

"If I say 9, you are going to say 8."

"That's right. I *think* that you know my secret." Thus the pupils have learned a way to demonstrate their discovery.

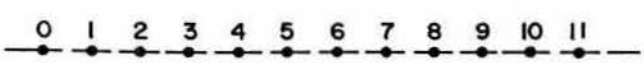
Many pupils are not as quick as Bill. So Miss Atkins may put on the board a record of the development thus far, first listing the number called out and then the reply:

$$5 \rightarrow 4$$
$$1 \rightarrow 0$$
$$7 \rightarrow 6$$
$$9 \rightarrow 8$$

The last item was Bill's contribution. Others might be:

$$6 \rightarrow 5$$
$$2 \rightarrow 1$$
$$4 \rightarrow 3$$

When the game is new or the rule difficult, a number line on the board will help:



"Let's act out the game we have played. When someone called out 5, I answered 4. Point to 5 on the number line. Keep your finger on that point. I answered by saying 4. Point to 4 on the line."

Each step is similary acted out.

"Now, if you call out 8, put your finger at the point on the number line."

$8 \rightarrow \_\_\_$

"What do you think I will reply?"

"Seven, of course!"

In some such way the idea can be made clear to even the slowest pupils. Care was taken not to talk about the teacher's rule itself—only about the results of applying it. Thus, every pupil must make the discovery for himself—some with minimum clues, others only after several different kinds of clues. There is a bit of artistry in giving clues only when needed and finding ways to reserve at least a little for even the slowest pupil to find out for himself.

Usually some pupil will note a curious fact about this rule.

"Suppose I call out zero. What will you answer?" a pupil may ask.

"Point to zero on the number line," the teacher suggests. "You can see that I would have to ask you to pick some other number." (Unless the teacher would like to talk about positive and negative numbers and zero.)

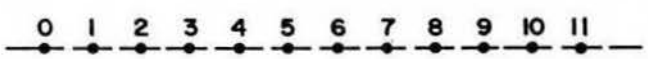
"I have a new rule. Call out a pair of numbers and I will reply by saying a number." Here is a record of a simple rule:

$2, 3 \rightarrow 5$
$4, 0 \rightarrow 4$
$3, 1 \rightarrow 4$
$1, 2 \rightarrow 3$
$1, 1 \rightarrow 2$
$3, 1 \rightarrow \_$

A more difficult rule is behind the following record:

$2, 4 \rightarrow 3$
$0, 2 \rightarrow 1$
$4, 0 \rightarrow 2$
$5, 9 \rightarrow 7$

This last rule may puzzle the pupils. Perhaps a number line can help shed light on the rule:



"If someone called out 2 and 4, I would reply with 3. Point to the numbers called out (2 and 4). I would reply with 3.

"If someone called out 4 and 0, I would reply with 2. Point to 4 and 0 on the number line. I would call out 2.

"Point to 5 and 9—and remember that I would answer 7. What would I answer if someone called out 1 and 7?"

$1, 7 \rightarrow \_\_\_$

"Four," Bill might answer, and quickly ask: "But what would you do if I called out 5 and 6 or if I called out 3 and 0?"

"I would have to ask you to pick another pair of numbers." (Unless the teacher is ready to discuss fractions.)

An unlimited number of rules can be devised on any level of sophistication. Further, there are many formats to be used in presenting rules, such as:

a. 1, 4, 7, 10, 13, , , . . .

b.

3	1	4	10	0	9	6
7	3	9	21	1		

c.

7	4	3	1
9	9	0	9
11	6	5	1
	8	6	
6			2

Mathematics is essentially a study of patterns. It has its own fascinating patterns and it is often employed to help reveal patterns in the physical world. When a scientist or any observer is searching for a better understanding of what he notices, he usually tries to set down a description of his observations. Whenever possible he will try to write down numbers that in some way help him discover relationships. He might record the following observations:

	1st	2nd	3rd	4th	5th
minutes	1	3	4	5	10
miles	1	9	16		

In his first three observations he notes time and distance. A pattern is strongly suggested. He may check his guess by making further checks for 5 minutes and 10 minutes. If his results are 25 miles and 100 miles, respectively, he will feel that he can formulate a rule or law that will suggest a pattern in certain real events ($1 \times 1 = 1$, $3 \times 3 = 9$, $4 \times 4 = 16$ . . .).

In mathematics we study patterns because they suggest basic principles of the subject itself and because mathematics helps record and explore patterns of the world we live in. The discovery of *one* basic principle can remove the necessity for learning many separate facts. Learning to discover patterns is a big step toward self-education. The recognition of the mathematical relationship among quantities in a scientific experiment allows further costly experimentation to be replaced by inexpensive computation.

Thus, this exercise or game of *What's My Rule?* has three important dimensions: (1) it can be used to motivate drill in the basic combinations of arithmetic; (2) it poses problems that pupils can think through to a solution; (3) it uses a method fundamental to the experimental sciences.

C. Provide for Individual Differences

As teachers, we strive to reach every pupil. We attempt to *intensify* the instructional process so that learning for all pupils, from the slowest to the

fastest, can proceed at an optimum level. Our strategy for this intensification begins with our concern for each pupil's *level*, *rate*, and *interest*. We further our aim as we choose activities and materials which, by their design, provide for the great variation of ability we find in our classes.

MATH WORKSHOP shares this goal with you—the intensification of the learning process for every pupil. In our concern about *level*, we have built *multi-level* materials into the program. To provide for differences in the learning rates of pupils, we have pioneered in developing problems in categories like *What Can You Say?* and *What's My Rule?*, which, by their nature, make it possible for the slowest pupil to progress while the most able pupil is inspired to investigate the questions more deeply. Our tests and suggestions for informal appraisal will aid you in making provision for the various levels and rates of your pupils. Further, we have developed problems and activities which may excite some pupils so much that they will ask whether they have to go to recess, or lunch, or some other coveted activity.

We have built MATH WORKSHOP with these ideas in mind. We have tested them on pupils for more than ten years. This present edition represents a third revision of the original material. The form of the present books, then, is a result of field testing with tens of thousands of pupils by the authors, their colleagues, and thousands of teachers from all parts of the country and in every conceivable type of school.

There are five specific ways in which we developed and organized MATH WORKSHOP so that its materials and suggestions would contribute to our common goal:

1. Through the Multilevel Structure of the Seven Basic Pupil Books

These Pupil Books were given the following designations: Beginner's Book, Level A, Level B, Level C, Level D, Level E, and Level F—so that at any grade a Pupil Book appropriate to the levels and rates of the pupils could be used. In general, the Beginner's Book would be used for Kindergarten or first grade, Level A for first grade, Level B for second grade, etc. However, a slow class or slow group in grade 3 might use Level A or B rather than Level C. Further, a very bright group may go on to Level D. Suggestions are made in the early pages of the Teacher's Guide to help you make this decision.

2. Through the Multilevel Structure of the Pupil Pages

Pages in the Pupil Book are designed with major consideration given to posing challenging questions. In the previous section we were able to demonstrate through examples how this approach to teaching provides for success for even the slowest pupil. At the same time, such a multi-dimensional technique opens the door to higher level responses and further investigations by the more able pupil.

3. Through the Multilevel Suggestions in the Teacher's Guide

Many suggestions are made throughout each Teacher's Guide for adjustments and extensions of activities to accommodate the various levels of achievement. Further, we should like to describe at this point four of the general ways in which individual pupil pages can be taught:

a. Use the page to conclude a development.
b. Use the page as a basis for discussion.
c. Let the pupils work in teams.
d. Let pupils work independently.

a. In those areas in which the teacher has many ideas of his own, there is no substitute for direct teaching—with liberal use of manipulative materials, the chalkboard, and class discussion. The pupil page is then used late in the development to tie the idea down or simply to record the progress that has been made toward deeper understanding.

There are limitless ways to initiate lively activities around important and useful concepts. MATH WORKSHOP provides many such activities for you to choose from.

After handling manipulative materials, playing number games, and other such preliminary work, pupils are ready to move through the pupil page in the final few minutes of the arithmetic period.

b. In contrast, some pupil pages might be gone through very carefully with an entire class. Such a page or sequence of pages offers many opportunities to dig deeply into "What's going on here?"

c. The team approach involves two or more pupils working on their own. It is very much like the fourth approach except for the rustle of activity as pupils speculate on theories as how to proceed. Teams may include pupils of many different levels of ability. This allows pupils to learn from one another. A contest based on the rate and accuracy of the team's effort is often a good motivational device.

d. A fourth approach might be called *Do-it-all-by-yourself*. Begin it by saying: Open your book to (name the page). Look it over. Think about it. See if you can decide for yourself what is to be done. Wait a while before you ask for help. And let's make today a "quiet day."

Often, the first question a pupil faces when he begins work on a pupil page is: What am I supposed to do? There are almost always sufficient clues provided so that with a bit of study pupils can discover the intent of the exercises. Traditional materials have provided too little incentive for pupils to figure out for themselves the nature of the business at hand.

Now you feign reluctance to help but are, of course, watching carefully for those who haven't sufficient self-confidence to get started on their own. If several pupils are unable to get started, suggest that they work in pairs, helping each other quietly. Enlarge these helper combinations if necessary. Grouping like this will usually get the activity under way for everyone. Let

time run out on the session if you wish, suggesting that those who haven't finished may wish to take the work home with them to finish.

4. Through the Tests and Suggestions for Informal Appraisal in the Pupil Book and Teacher's Guide

The first ten practice sheets in Level E are designed to serve as an Informal Arithmetic Inventory that will aid you in identifying the levels of pupils and that will suggest where further stress will be needed in teaching. The Teacher's Guide offers many suggestions for interpreting these tests.

5. Through an Index to Remedial Activities in Earlier Levels of MATH WORKSHOP

An index is provided in this guide indicating the pages in Levels A, B, C, and D that may be used for remedial work.

6. Through Supplementary Materials Developed by the Authors of MATH WORKSHOP

Described in Section IV below are a number of MATH WORKSHOP supplementary multilevel teaching aids which are available from Encyclopaedia Britannica Press. These include: (a) a teacher training text, *Discovery in Elementary School Mathematics;* (b) Practice Sheets; (c) MATH WORKSHOP Supplementary Activities; and (d) Games and Enrichment Activities.

Nowhere today has one program collected, invented, and brought together into a comprehensive form such a great variety of material aimed at the intensification of the learning process.

D. Use the Discovery Method

As we have noted repeatedly, each page in MATH WORKSHOP, the activities suggested in the Teacher's Guide, the practice sheets, and other aids are designed so that there are several levels on which pupils can participate—including opportunities for discovery and success for even the slowest learners.

The *What Can You Say?* type of activities we present are truly open-ended discovery exercises with challenge for even the most advanced and able pupils in the class. While some questions can be easily settled, more will have been left open. Some pupils will wish to explore these open questions on their own. Such pupils ought to learn to work independently, sparing the teacher's time for those who need more of it. Actually, what the more able pupils need most is encouragement and the opportunity to develop a taste for independent study. The highest level of learning has been reached when a pupil begins to pose problems of his own and to work toward their solution.

In MATH WORKSHOP there is extensive provision for introducing or practicing the basic facts of arithmetic and for solving problems. In most cases such practice is organized so that it leads to the emergence of a pattern or the solution of a problem.

Discovering such patterns is one of the most rewarding experiences in mathematics. It is called the "Aha!" or the "Gee-Whiz!" experience—a gasp, the light dawns! "I see it!" These are events for which teachers work long and hard. One of the most disheartening facts teachers must face is that some pupils simply "don't see it." Yet this is also a challenge—it calls for exercising skill in teaching. Slow learners need more clues—more obvious clues; but the finest teacher will reserve a wee bit for each pupil to "see" for himself. If the teacher's best efforts fail, he may have to give away the secret and try again next time. This is an art that does not come easily, but the eventual results are deeply rewarding.

When the results of work in arithmetic exhibit an unusual regularity, the inquiring mind looks for an explanation. One problem may be solved but others are suggested. This is an exciting characteristic of mathematics.

When pupils discover a relationship, they are excited and eager to stay with that discovery. They will readily put this new idea to work, showing more examples of *it*. They come alive, often with an "Oo-o-o! I've got it," or with a burst of industry using a newly-discovered relationship to reel off answers with speed and confidence.

What is the *it* they have glimpsed? Often, if we distract their attention with a request for them to *tell about it* or *put it in words*, we have selected a method that could not be better calculated to turn pupils against discovery.

That a pupil can move ahead by putting *it* to work successfully is certainly all the evidence we need in order to know that he *has it*. Our only justification for veering off in the direction of a word-pulling exercise is a feeling that *it* is a generalization that ought to be verbalized in order to be usable. But research does not support this. Transfer to future experiences is not necessarily increased by attempting to *verbalize a generalization*. Subvocal or nonverbal awareness, the new-found ability to move ahead more speedily, more directly, more confidently, is our main goal.

When pupils experience an insight—when they leap ahead because they have discovered a pattern—we ought to be very careful about letting words into the act too soon. The pupils probably do not have the language needed to express their discovery in words. If they try, we must, of course, help them. But too often such help and explanation degenerate into hair-splitting, an activity far removed from the act of discovery.

Gertrude Hendrix of the University of Illinois Committee on School Mathematics, who has done much research on discovery in the teaching of mathematics, warns that imprecise verbalization may *mutilate* a discovery. On the other hand, when some pupils (and teachers) make discoveries, they wish others to know about it. But herein lies a danger. Whenever a person reveals his secret to one who has not been as fortunate, the latter is deprived of the delight of discovering it for himself.

In MATH WORKSHOP we invent ways in which pupils can share their discovery without revealing the secret. We have them whisper their new idea in the teacher's ear; we encourage them to make up additional examples which will provide further clues for those who haven't found a pattern that underlies several examples. But we urge children to keep the secret—for a while at least.

Once in a while we find pupils who have not invented shortcuts for solving certain kinds of problems. For example, suppose that pupils use counting by ones in addition problems which involve fairly large sums. Hopefully, such pupils will become discouraged and begin to look for easier ways of determining the sum, such as counting by fives. If such easier ways are discovered, the pupils should be highly praised and urged to demonstrate these techniques to the entire class. If these examples are not sufficient to shake all pupils loose from counting by ones, then you must step in and urge the conservative ones to drop the slower technique. Please temper the discovery method with common sense.

E. Use Representational Materials

When mathematics loses contact with the world of things and with the events out of which it came, the subject becomes mysterious to most pupils. If they do not understand that in mathematics we are discussing the way things happen as we manipulate objects, pupils must resort to rote memory. MATH WORKSHOP activities stress the fact that arithmetic provides a means of recording or reporting what happens in the physical world.

Many kinds of manipulative or representational materials are suggested and discussed in this Teacher's Guide. These include various types of counting materials, geometric shapes, number lines, etc. Perhaps the most accessible source of representational materials is the pupil pages themselves. We believe that all of elementary mathematics can be abstracted out of experiences in the physical world. The very word *workshop* in the title of our program indicates our belief in the value of learning mathematics by working with physical objects. In some cases, the pupil pages are the workshop—pupils work with the two-dimensional pictorial representations of physical objects and apparatus and use the forms provided to keep records. Frequently the teacher will decide that the pupils need to work with three-dimensional equipment. In those cases, the teacher may provide equipment like that pictured on the pupil pages or like that suggested in the Teacher's Guide.

F. Teach for Mastery of Basic Facts and Algorithms

Mathematics provides a language to help report events that happen in the world of things. As we have noted, the core of elementary school mathematics

is counting. Because this process becomes tedious when many things must be counted, shortcuts to counting have been invented and improved. Thus, elementary school mathematics is essentially concerned with counting and with shortcuts to counting.

To some who have not studied the statements that precede this section, many MATH WORKSHOP activities appear to be highly motivated exercises to provide practice in the basic facts, combinations, and processes of arithmetic. In the vernacular, they wear the garb of "sugar-coated drill." Actually, they are valid educational devices, because they help establish that automatic recall of fundamentals without which pupils are mathematical stutterers.

Every teacher knows that many pupils are mathematical stutterers because they can't easily recall the basic facts, and stumble over the algorithms. They hesitate and flounder; the discussion bogs down; they lose the argument; they miss the feeling of movement toward a solution; they cannot enjoy mathematics.

This area of trouble is so universally known that it is surprising so many are only casually concerned! There is much hue and cry to bring new and unusual content into the elementary school curriculum—numeration using bases other than ten, sets, properties of numbers, and other concepts—yet there seems to be silence about such a crucial matter as helping pupils master the fundamental combinations. Without this mastery their progress toward understanding mathematical ideas is very painful.

While we have emphasized the scope of our program in terms of nine unifying ideas, *the basic facts and algorithms of arithmetic are taught and retaught throughout the program in all basic and supplementary materials.*

IV. MATH WORKSHOP TEACHING AIDS

A. Teacher's Guide

The introduction presents a point of view and a rationale for our approach to teaching elementary school mathematics. You will find this useful in interpreting many of the things we have included in the pupil pages. You will also find it useful in explaining the program to parents.

Each pupil page is reproduced in the Teacher's Guide in full size and in full color with answers printed in red.

Each pupil page is accompanied by a facing guide page that is divided into sections. Let's consider these sections:

1. *Tasks.* This section contains a very brief description of what the pupil will be doing as he works on the pupil page. We suggest that you study the pupil page in a pupil's book and work on it before you read our description and the answers we give.

2. *Purpose.* Here we describe what a pupil is intended to learn as he works on the pupil page or as he carries out some of the activities described in The Lesson. (Naturally, our description contains a conservative estimate of what might happen. Mathematics is full of surprises and so are pupils.) If you are aware of our purpose, you will find it easier to help pupils achieve it.

3. *Unifying Ideas.* In this section we list those of the nine unifying ideas mentioned on pages 10-11 to which the exercises on the pupil page and recommendations in the notes on the page in the Teacher's Guide can be expected to contribute. Please regard the list as a minimum statement, and be prepared for pupils who may wish to engage in activities that go beyond what we have described for the page.

4. *The Lesson.* In this section we describe one or more of the ways we might build a lesson around the pupil page. No teacher could possibly find time to use all our suggestions in any school year. Choose the ones which best suit your style of teaching, and vary your choices from year to year.

B. Practice Sheets

On each of the first 96 pages of the Level E Teacher's Guide, a practice sheet is presented (in reduced size) and discussed in the notes when the purpose and use of the sheeet are not self-evident. These pages may be copied or adapted by the teacher. (They are also available in full size from Encyclopaedia Britannica Press.)

Every practice sheet contains the beginning or continuation of the investigation of a problem, the exploration of a mathematical situation. Generally, the activity is a different approach to the development found on the corresponding pupil page, an extension of it, or a self-contained activity that provides extensive practice. No page is just random drill. The bottom section of most practice sheets provides story problems and *What Can You Say?* problems that will challenge some of the most able pupils in any fourth grade class. Do not expect all pupils who try these problems to get them right without help.

C. Math Workshop Supplementary Activities

A package of supplementary activity worksheets is available for every level, A through F, from Encyclopaedia Britannica Press. Each package includes 32 copies of 64 different activity pages. These can be used independently by pupils for seatwork and homework. Able pupils might be given supplementary activity materials from advanced levels. Slow pupils might work on these materials from lower levels. Each set of supplementary activities includes a Teacher's Manual with brief suggestions for each page and an answer key.

D. Discovery in Elementary School Mathematics

This is a professional development textbook that suggests to teachers many ideas for teaching mathematics at every grade level. This book is divided into two major sections—Multilevel Research Activities and Multilevel Worksheet Activities.

Multilevel Research Activities present 27 exploratory experiences which can be introduced on the chalkboard and with simple teacher-made manipulative materials at any grade level. Each activity starts simply and spirals upward with extensions and elaborations that will appeal to the more mature and sophisticated mind. Each activity focuses on mathematical structure in a context of discovery learning.

The Multilevel Worksheet Activities section contains 24 units, each of which explores some of the unifying ideas of mathematics. Each unit begins with a brief introductory statement regarding mathematical structure and discovery. Each unit includes six activity worksheets, scaled in difficulty so that a teacher can present the ideas discussed in the unit to pupils at any level. These pages may be copied or modified by teachers for classroom use.

E. Games and Enrichment Activities

We have prepared a manual on the use of representational materials, such as Cuisenaire®* rods and Geo-boards. These materials can be used as the basis for games and other enrichment activities, many of which are mentioned or introduced in this Teacher's Guide. They contain many unusual applications of the concepts considered in the pupil pages.

F. Tests in the Pupil Book

Scattered throughout MATH WORKSHOP, Level E, are pages that we suggest you use as tests. These are pages 34, 35, 80, and 85.

G. Informal Arithmetic Inventory

The first ten practice sheets in Level E are designed to serve as an informal inventory. Teachers have used such informal tests in reading and spelling to provide each pupil with material at his instructional level. You might wish to do the same thing in arithmetic.

The psychological principle of success in learning is the basic notion behind such an inventory or placement test. We all know that the ideal learning

*Cuisenaire—Registered trademark of Cuisenaire Company of America, Inc.

environment provides for heaping measures of success and challenge, and eliminates or minimizes frustration. Unfortunately, no matter how carefully textbooks are designed—including MATH WORKSHOP, which provides as much help as possible for pupils at different levels of ability—some pupils are frustrated unless they are able to work in easier books in the series. It follows too that some few should work in harder books if they are to be sufficiently challenged.

You may wish to initiate this year's program by having pupils complete the first ten practice sheets first. In this case keep a record of their percentage of accuracy on each page for the problems you have assigned. You will then be able to determine whether Level E is the appropriate text for each pupil or if a lower or higher level in the series is called for.

You will notice that we are measuring more than computational skill in these ten pages. Consistent with our promise, we place computational problems most often in a context which requires more than automatic recall. You will be able to learn what each pupil does when confronted by a situation with which he is not familiar. In this way you will have a basis for knowing how pupils go about solving new problems.

We offer the following as a guide to the placement of your pupils as you observe them working on these ten pages: (1) if a pupil scores 0-50% on the series, he might be placed in the Level C pupil book; (2) if a pupil scores 51-70%, he might be placed in the Level D pupil book; (3) if a pupil scores 71-100%, he can begin in the Level E pupil book.

As with any evaluation technique for young pupils, care must be taken not to put more faith in tests than they deserve. Your own judgment of a pupil based on your face-to-face work with him is probably more reliable than scores on tests.

This brings up the general problem of marking or grouping pupils as they work on all of the pupil pages. The pupil pages should be considered opportunities for investigation and exploration on the part of the pupil rather than as pages to mark for a grade. There are many pages on which pupils are not expected to get 100% on the first try. Nothing would be more damaging to their regard for mathematical inquiry than to be pummeled with red marks and low grades. We suggest that you *avoid* marking pages with a *minus* score. The number right or the percentage of accuracy is a more positive approach. By all means, keep a light touch rather than a heavy hand when checking and marking these pages.

V. PARENT EDUCATION AND INVOLVEMENT

Ill-advised and misleading publicity in newspapers and magazines has frightened parents into believing they can no longer help their children at home. The practice sheets may help considerably to soothe these parents. No new mathematical language is used on any of these pages. Most require extensive

mental or scratch-paper computations in addition to the results finally recorded on the sheets. None of these computations is beyond the experience of most parents.

A common cause of misunderstanding between parents and children is concern for doing a computation the "right way." The child will often argue that the book's way or the teacher's way is the "right way"—and the parent will prefer his way. Of course, there is no one right way. There are only right answers or right results or valid conclusions. The way one chooses is his own business, a matter of personal taste—so long as it does not lead him to nonsense. MATH WORKSHOP suggests many approaches to problems and urges teachers to place a high premium on the discovery of many ways to find a right answer. We have argued elsewhere that if a child does not have at least one alternate method of finding a right answer, then he does not understand the method he is using.

Parents can do much to help children move ahead in mathematics. If they work together with their children toward solutions of problems, avoiding questions of right and wrong methods, mathematics can become a common interest, contributing to a more constructive home environment. Parents may even come to realize that the fact that they "can't even keep the checkbook straight" does not really indicate a lack of mathematical talent or prevent them from participating with enjoyment in mathematical investigations. In fact, in these activities, parents are likely to discover that mathematics is a part of their culture that was denied them by an early contact with a crude facsimile of the subject.

Practice sheet activities provide opportunities for children and parents to work together and to think together about problems—a much more productive experience than completing assigned tasks in a prescribed way. Suppose a question is brought home: How many ways are there to make change for a quarter? Out come the coins from father's pocket or mother's purse or the piggy bank. Someone must keep a record. How can you be sure that you have all the combinations? What kind of record will help answer that last question?

"Let's try a dollar next!"

And later, in the car, "I'm thinking of a combination of 9 coins that add up to 29¢. What coins am I thinking about?"

Practice sheets were designed with "home consumption" in mind. They may help bridge the gap that misinformation and rigid assignments create. Of course, the Supplementary Activities can also be used in the same way.

VI. PUNCTUATION AND ORDER OF OPERATIONS

In Level E pupils will sometimes face expressions such as "$8 - 2 + 3$," which appear to be ambiguous. Does this expression stand for 9 or for 3? That is, should one subtract 2 from 8 and then add 3 to the result, or should one add

3 to 2 and subtract the result from 8? One way to remove the ambiguity is to punctuate the expression with a pair of parentheses. Thus,

$$(8-2)+3=6+3=9$$

and

$$8-(2+3)=8-5=3.$$

A second way to remove the ambiguity is to establish a convention or agreement. The convention we follow in MATH WORKSHOP is that when an expression involves addition and subtraction, one should proceed from left to right in carrying out the indicated operations, unless parentheses are used to indicate a contrary order. Thus,

$$8-2+3=6+3=9$$

and

$$9+4-1=13-1=12$$

and

$$7-2-1-3=5-1-3=4-3=1.$$

When an expression involves multiplication and addition or subtraction, the convention we follow says that, unless parentheses indicate a contrary order, one should perform the multiplications first, and then the additions or subtractions from left to right. Thus,

$$2\times3-1=6-1=5,$$

$$4+3\times2=4+6=10,$$

$$3\times5+7\times5=15+35=50,$$

$$\begin{aligned}4\times2+5\times3-6\times2&=8+15-12\\&=23-12\\&=11.\end{aligned}$$

VII. A FINAL WORD

Makers of I.Q. tests search for measures of the pupil's ability to think. They try to rule out questions that merely assess achievement in subject-matter areas. Therefore, they would not ask pupils to complete the following familiar examples from arithmetic. These questions would more likely appear in an achievement test.

$$\begin{array}{r}3\\+7\\\hline\end{array} \qquad \begin{array}{r}41\\-16\\\hline\end{array} \qquad \begin{array}{r}346\\\times\ 27\\\hline\end{array} \qquad 5\overline{)135} \qquad 8\times3=$$

$$\frac{1}{2}+\frac{2}{3}= \qquad\qquad 1.19=17\%\times\_\_$$

For measurement of I.Q. the questions from mathematics would be more like these:

I. Extend each of the following sequences:

a. 2, 5, 8, 11, 14, ___, ___, ___ .....
b. 1, 2, 4, 8, 16, ___, ___, ___ .....
c. 3, 4, 6, 9, 13, ___, ___, ___ .....
d. 1, 1, 2, 3, 5, ___, ___, ___ .....

II. Fill the blanks in the following examples:

a.

6	3	1	4	2	
(10)	(7)	(5)	( )	( )	(11)

b.

2	4	3	1	5	
↓	↓	↓	↓	↓	↓
4	16	9	___	___	0

c.

1, 3 → 8	3, 2 → 10
5, 0 → 10	6, 3 → ___
2, 5 → 14	, 4 → 12

III. Circle the number that seems most out of place in each row:

a. 6, 10, 2, 7, 8, 4
b. 25, 70, 43, 16, 31, 52
c. 13, 15, 19, 3, 11, 7

Such problems, requiring a familiarity with arithmetic, are held to be more appropriate as measures of the ability to think. They are designed to test intelligence rather than achievement.

Mathematics is a fertile source of I.Q. test questions because mathematics is fundamentally a study of patterns and relationships. Many of the questions selected are similar to the examples given above. Behind that choice is the test-maker's estimate that pupils have not already explored and investigated such questions.

What an indictment of the elementary school curriculum in mathematics!

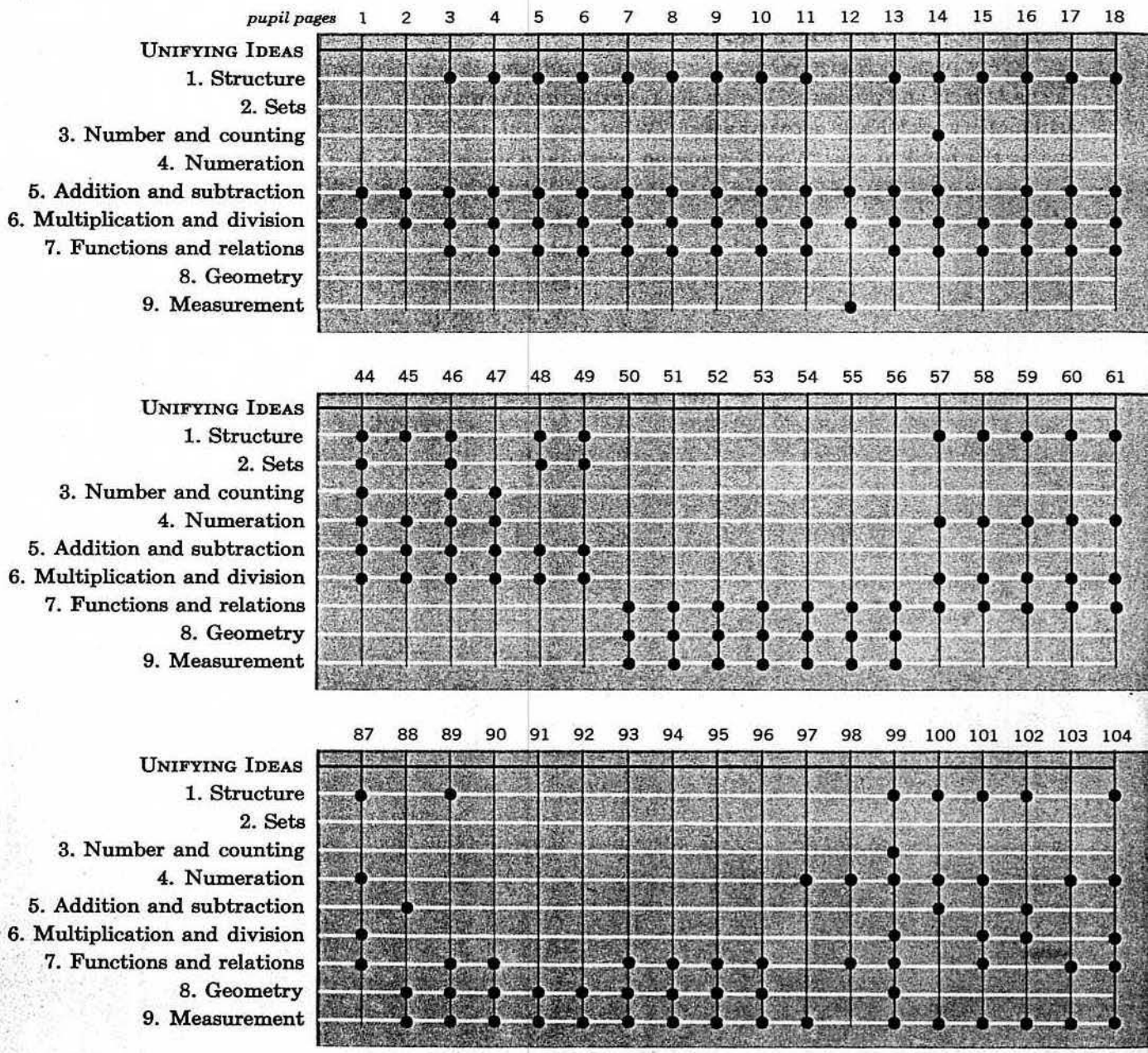
If arithmetic is limited to memory of number combinations and rules for manipulation, then it is a dead, dull, dreary task against which an active,

eager, creative mind must surely rebel. However, if we broaden our thinking and look everywhere for patterns and relationships, the subject stimulates the mind to inquire and search out, and discover!

Every page in MATH WORKSHOP contains problems that probe beneath the basic facts and processes. We are always exploring the question: Why does arithmetic work? Each activity asks: "What can you say?" Each activity invites the pupil to ask: What's going on here?—and urges him to find out.

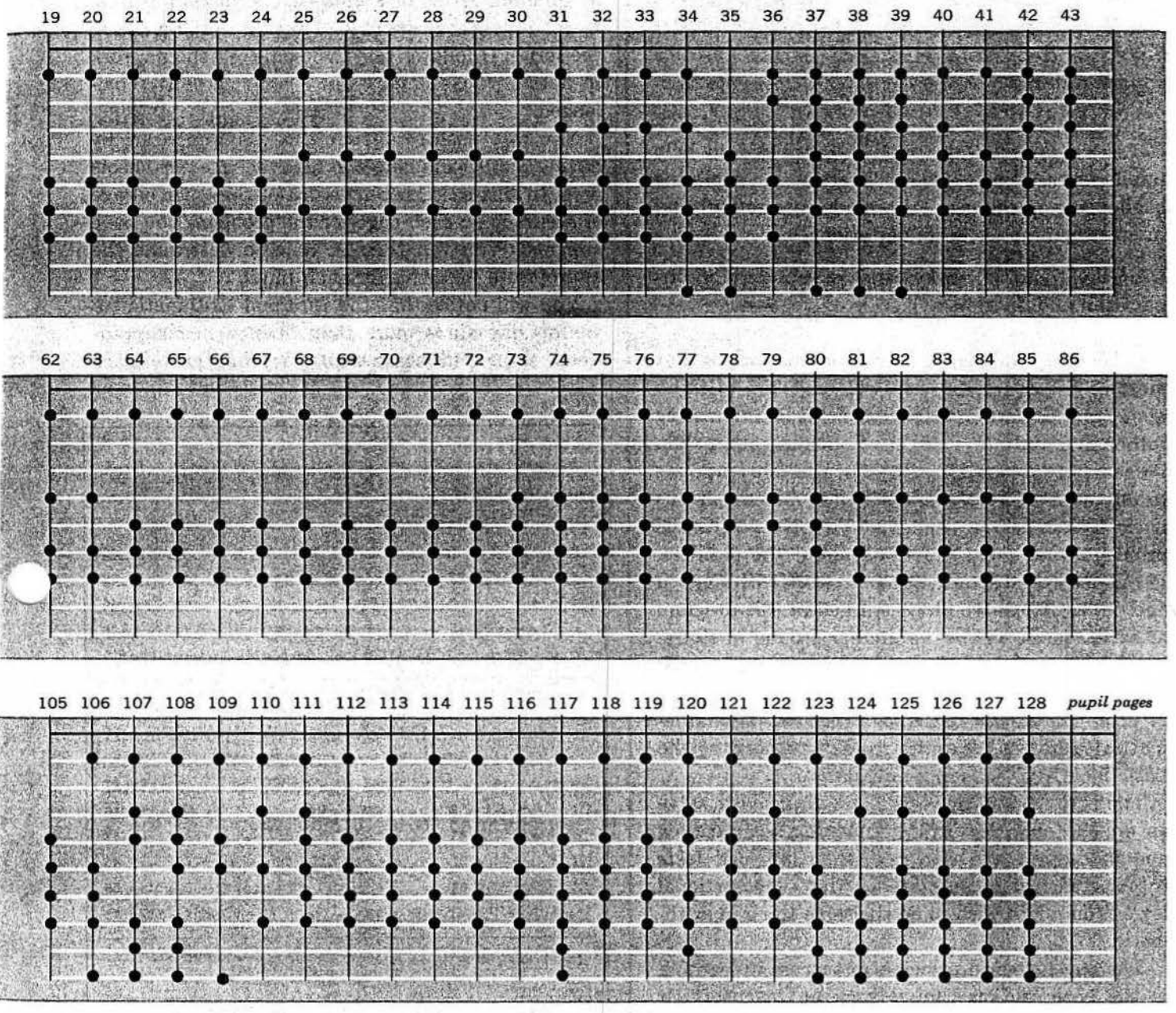
Robert W. Wirtz
Morton Botel
Max Beberman
W. W. Sawyer

Level E Revised



Consider each unifying idea . . . on how many pages will you find it

1. Structure	103 pages
2. Sets	10 pages
3. Number and Counting	27 pages
4. Numeration	62 pages
5. Addition and Subtraction	83 pages
6. Multiplication and Division	102 pages
7. Functions and Relations	96 pages
8. Geometry	27 pages
9. Measurement	40 pages



Consider each page . . . how many unifying ideas will you find on it?

6 pages contain two unifying ideas,
25 pages contain three unifying ideas,
54 pages contain four unifying ideas,
22 pages contain five unifying ideas,
11 pages contain six unifying ideas,
10 pages contain seven unifying ideas.

A Word from the Authors to the Readers

Here is the kind of question we promise *not* to ask you very often:

> Arlene made a 75¢ purchase.
> She gave the clerk a dollar bill.
> How much change should she get?

That's an uninteresting question. All Arlene needs to know is that she should receive a quarter's worth of change.

In this book you are more likely to find questions like this:

> How many different combinations of pennies, nickels, and dimes make a quarter's worth of change?

You probably don't know the answer at once. Almost no one does. Almost anyone can answer the question if he will think about it, do a little planning, and carry out the plan using pencil and paper when necessary.

Someone might start a list of combinations such as:

> 5 nickels
> 1 dime, 1 nickel, 10 pennies
> 2 dimes and 5 pennies
> 1 nickel and 20 pennies

Such an approach could quickly lead him into trouble. How will he know whether or not he has listed *all* possible combinations?

Planning Ahead

Trouble can be avoided by working out a plan to list all possible combinations.

Here is an example of such a plan:

A. The combinations which include 2 dimes:

2 dimes,	1 nickel,	0 pennies
2 dimes,	0 nickels,	5 pennies

B. The combinations which include just 1 dime:

1 dime,	3 nickels,	0 pennies
1 dime,	2 nickels,	5 pennies
1 dime,	1 nickel,	10 pennies
1 dime,	0 nickels,	15 pennies

C. The combinations which include zero dimes:

0 dimes,	5 nickels,	0 pennies
0 dimes,	4 nickels,	5 pennies
0 dimes,	3 nickels,	10 pennies
0 dimes,	2 nickels,	15 pennies
0 dimes,	1 nickel,	20 pennies
0 dimes,	0 nickels,	25 pennies

There are exactly 12 different combinations of dimes, nickels, and pennies that make a quarter's worth of change.

Notice the following:

1. We thought about the problem.
2. We devised a plan to solve the problem.
3. We carried out the plan using paper and pencil when necessary.
4. We turned up new information which helped us solve the problem.
5. We can always use this plan to solve problems like this one.

Tasks: Pupils study pupil page 1 to learn how to explore a challenging question. On pupil page 2 pupils consider ways of making change for a quarter and then complete a table recording all possible combinations. They then consider all possible combinations for 24¢, 26¢, and $5.

Purpose: To cultivate pupil interest and to help pupils learn how to plan a strategy for exploring challenging questions. To provide meaningful and interesting practice in computation.

Unifying Ideas: Addition and subtraction; multiplication and division.

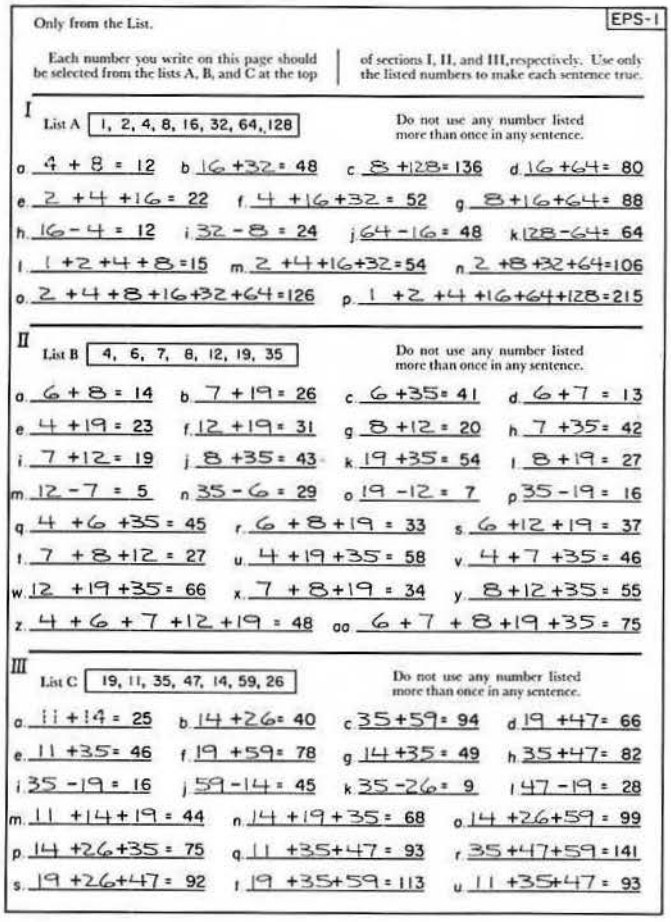
The Lesson: Pupil pages 1 and 2 promise pupils that most of the questions they will be asked to think about and explore in this course will be interesting rather than ordinary; less than obvious rather than immediately evident. It will not be possible for pupils to rely on memory alone in solving problems in this book. Herein lies the challenge. To solve worthwhile problems they must (1) think about the problem, (2) plan a line of attack, (3) go to work with a pencil and paper. Those who have worked in this program at earlier levels will be on familiar ground in studying this kind of mathematics. For pupils new to the program, a period of adjustment may be needed, especially if they have the idea that solving mathematical problems requires merely memory of the basic facts and skills in manipulation.

The first ten practice sheets are designed to serve as an *Informal Arithmetic Inventory*. In reading and in spelling we have used such informal tests to place each pupil in material at his instructional level. You might wish to do the same thing in arithmetic.

The psychological principle of success in learning is the basic notion behind such an inventory or placement test. We all know that the ideal learning environment provides for heaping measures of success and challenge and eliminates or minimizes frustration. Unfortunately, no matter how carefully textbooks are designed—even MATH WORKSHOP, which provides as much help as possible for pupils at different levels of ability—some pupils are frustrated unless they are able to work in easier books in the series. It follows, too, that some few should work in harder books if they are to be sufficiently challenged.

You may wish to initiate this year's program by having pupils complete the first ten practice sheets first. In this case keep a record of their percentage of accuracy on each page for the problems you have assigned. You will then be able to determine the appropriate textbook for each pupil.

You will notice that the Informal Arithmetic Inventory measures more than fluency in computational skill. Consistent with our promise, we place



Only from the List. EPS-1

Each number you write on this page should be selected from the lists A, B, and C at the top of sections I, II, and III, respectively. Use only the listed numbers to make each sentence true.

I List A [1, 2, 4, 8, 16, 32, 64, 128] Do not use any number listed more than once in any sentence.

a. 4 + 8 = 12 b. 16 + 32 = 48 c. 8 + 128 = 136 d. 16 + 64 = 80
e. 2 + 4 + 16 = 22 f. 4 + 16 + 32 = 52 g. 8 + 16 + 64 = 88
h. 16 - 4 = 12 i. 32 - 8 = 24 j. 64 - 16 = 48 k. 128 - 64 = 64
l. 1 + 2 + 4 + 8 = 15 m. 2 + 4 + 16 + 32 = 54 n. 2 + 8 + 32 + 64 = 106
o. 2 + 4 + 8 + 16 + 32 + 64 = 126 p. 1 + 2 + 4 + 16 + 64 + 128 = 215

II List B [4, 6, 7, 8, 12, 19, 35] Do not use any number listed more than once in any sentence.

a. 6 + 8 = 14 b. 7 + 19 = 26 c. 6 + 35 = 41 d. 6 + 7 = 13
e. 4 + 19 = 23 f. 12 + 19 = 31 g. 8 + 12 = 20 h. 7 + 35 = 42
i. 7 + 12 = 19 j. 8 + 35 = 43 k. 19 + 35 = 54 l. 8 + 19 = 27
m. 12 - 7 = 5 n. 35 - 6 = 29 o. 19 - 12 = 7 p. 35 - 19 = 16
q. 4 + 6 + 35 = 45 r. 6 + 8 + 19 = 33 s. 6 + 12 + 19 = 37
t. 7 + 8 + 12 = 27 u. 4 + 19 + 35 = 58 v. 4 + 7 + 35 = 46
w. 12 + 19 + 35 = 66 x. 7 + 8 + 19 = 34 y. 8 + 12 + 35 = 55
z. 4 + 6 + 7 + 12 + 19 = 48 oo. 6 + 7 + 8 + 19 + 35 = 75

III List C [19, 11, 35, 47, 14, 59, 26] Do not use any number listed more than once in any sentence.

a. 11 + 14 = 25 b. 14 + 26 = 40 c. 35 + 59 = 94 d. 19 + 47 = 66
e. 11 + 35 = 46 f. 19 + 59 = 78 g. 14 + 35 = 49 h. 35 + 47 = 82
i. 35 - 19 = 16 j. 59 - 14 = 45 k. 35 - 26 = 9 l. 47 - 19 = 28
m. 11 + 14 + 19 = 44 n. 14 + 19 + 35 = 68 o. 14 + 26 + 59 = 99
p. 14 + 26 + 35 = 75 q. 11 + 35 + 47 = 93 r. 35 + 47 + 59 = 141
s. 19 + 26 + 47 = 92 t. 19 + 35 + 59 = 113 u. 11 + 35 + 47 = 93

computational problems most often in a context which requires more than automatic recall. You will be able to learn what each pupil does when confronted by a situation with which he is not familiar. In this way you will have a basis for knowing how pupils go about solving new problems.

The following table offers a guide to the placement of your pupils as you observe them working on these ten pages.

Informal Arithmetic Inventory

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practice sheet 1 to top half of 7	0-50%	Level C Worktext
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Do not hesitate to introduce each type of problem on these pages on the board as readiness for independent work. Further, give a bit of help to pupils who are not sure of themselves.

Remember that this is not a standardized test but an informal test to determine which text in MATH WORKSHOP is appropriate to the pupil's instructional level—the level at which he can work with success and challenge.

Rather than working through an entire lower-level textbook with slower pupils, you may wish to select particular activities to correct deficiencies made evident by this *Informal Arithmetic Inventory*. Or, you may wish simply to move more slowly through Level E with such pupils and to give them more direction than the rest of the class. If pupils are simply unsure rather than in need of an extended review, the early portions of Level E provide ample experience for them to develop a firm grasp of the multiplication and division algorithms.

Pupils who score 70-100% on practice sheets 7 through 10 might be allowed to work through the problems in Level E independently and at their own rate. Certainly they will wish to study the special algebra and geometry sections in this level.

On practice sheet 1 the pupil must find numbers in the list that convert open sentences into true ones. Here is such an open sentence:

$$\underline{\quad} + \underline{\quad} = 48$$

Since 16 and 32 are in list A, this presents no great problem. However, the questions become more difficult and may require considerable trial and error such as asking pupils to find three numbers in list C whose sum is 48:

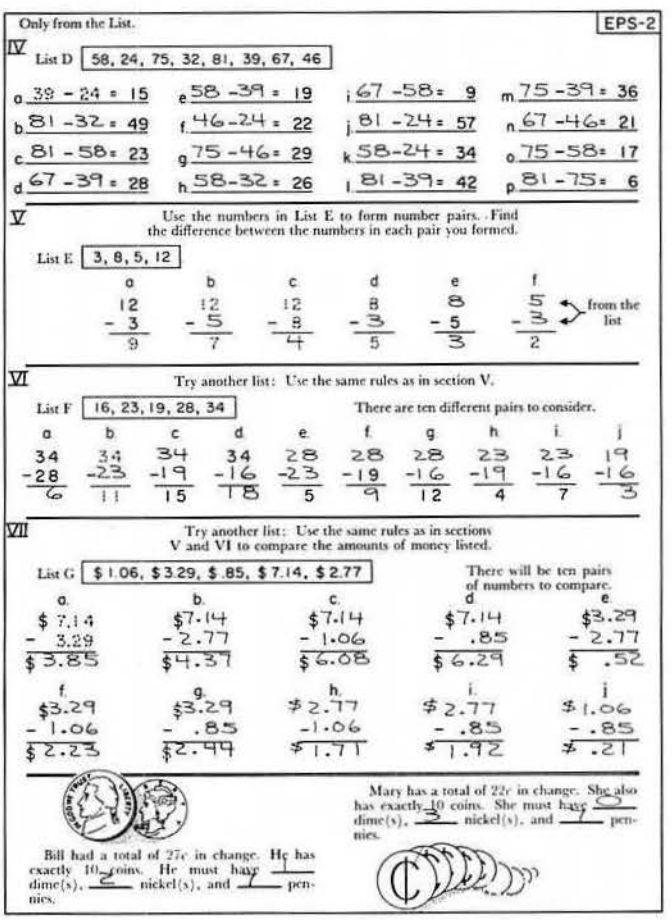
$$\underline{\quad} + \underline{\quad} + \underline{\quad} = 48$$

Practice sheet 2 builds on the *Only from the List* idea by asking for differences as follows:

From list D find two numbers whose difference is 19:

$$\underline{\quad} - \underline{\quad} = 19$$

Pupils have the further task on this page of comparing all pairs in lists E, F, and G. Finding all pairs requires more than casual consideration.



Only from the List. EPS-2

IV List D: 58, 24, 75, 32, 81, 39, 67, 46

a 39 - 24 = 15 e 58 - 39 = 19 i 67 - 58 = 9 m 75 - 39 = 36
b 81 - 32 = 49 f 46 - 24 = 22 j 81 - 24 = 57 n 67 - 46 = 21
c 81 - 58 = 23 g 75 - 46 = 29 k 58 - 24 = 34 o 75 - 58 = 17
d 67 - 39 = 28 h 58 - 32 = 26 l 81 - 39 = 42 p 81 - 75 = 6

V Use the numbers in List E to form number pairs. Find the difference between the numbers in each pair you formed.

List E: 3, 8, 5, 12

a	b	c	d	e	f
12	12	12	8	8	5
- 3	- 5	- 8	- 3	- 5	- 3
9	7	4	5	3	2

from the list

VI Try another list: Use the same rules as in section V.

List F: 16, 23, 19, 28, 34 — There are ten different pairs to consider.

a	b	c	d	e	f	g	h	i	j
34	34	34	34	28	28	28	23	23	19
-28	-23	-19	-16	-23	-19	-16	-19	-16	-16
6	11	15	18	5	9	12	4	7	3

VII Try another list: Use the same rules as in sections V and VI to compare the amounts of money listed.

List G: $1.06, $3.29, $.85, $7.14, $2.77 — There will be ten pairs of numbers to compare.

a	b	c	d	e
$7.14	$7.14	$7.14	$7.14	$3.29
- 3.29	- 2.77	- 1.06	- .85	- 2.77
$3.85	$4.37	$6.08	$6.29	$ .52

f	g	h	i	j
$3.29	$3.29	$2.77	$2.77	$1.06
- 1.06	- .85	-1.06	- .85	- .85
$2.23	$2.44	$1.71	$1.92	$ .21

Bill had a total of 27¢ in change. He has exactly 10 coins. He must have 1 dime(s), 2 nickel(s), and 7 pennies.

Mary has a total of 22¢ in change. She also has exactly 10 coins. She must have 0 dime(s), 3 nickel(s), and 7 pennies.

Perhaps some of you are curious enough to raise new questions about the problem we solved.

For example: Do any of the combinations have the same number of coins?

Again, you don't know, but you can find out. Let's put all combinations in a table, and note the total number of coins in each combination:

D — number of dimes

N — number of nickels

P — number of pennies

C — number of coins

D	2	2	1	1	1	1	0	0	0	0	0	0
N	1	0	3	2	1	0	5	4	3	2	1	0
P	0	5	0	5	10	15	0	5	10	15	20	25
C	3	7	4	8	12	16	5	9	13	17	21	25

We've started the table. Please complete it.

Do any of the combinations have the same number of coins?

Perhaps you would like to think about two more questions:

How many different combinations of dimes, nickels, and pennies are worth 24¢? 9

How many different combinations of dimes, nickels, and pennies are worth 26¢? 12

You may be able to find the answers to the last 2 questions by simply studying the table of combinations of coins that are worth 25¢. You may decide to make a chart to help you answer each question.

The point we are trying to make is this: You can devise a plan which can help you to solve the problem for yourself.

You may wish to explore other problems such as:

How many different combinations of common coins are worth five dollars?

By "common coins" we mean silver dollars, half-dollars, quarters, dimes, nickels, and pennies.

The authors have never tried to solve this problem. If you do solve it, perhaps you will send us your solution.

Thumb through the book you have in your hand. It contains many questions you will want to solve on your own or with a little help from others.

Try to solve this one:

Helen had seven common coins.

Think about this situation.

What can you say that isn't at first obvious?

Notes on Pupil Pages 1 and 2

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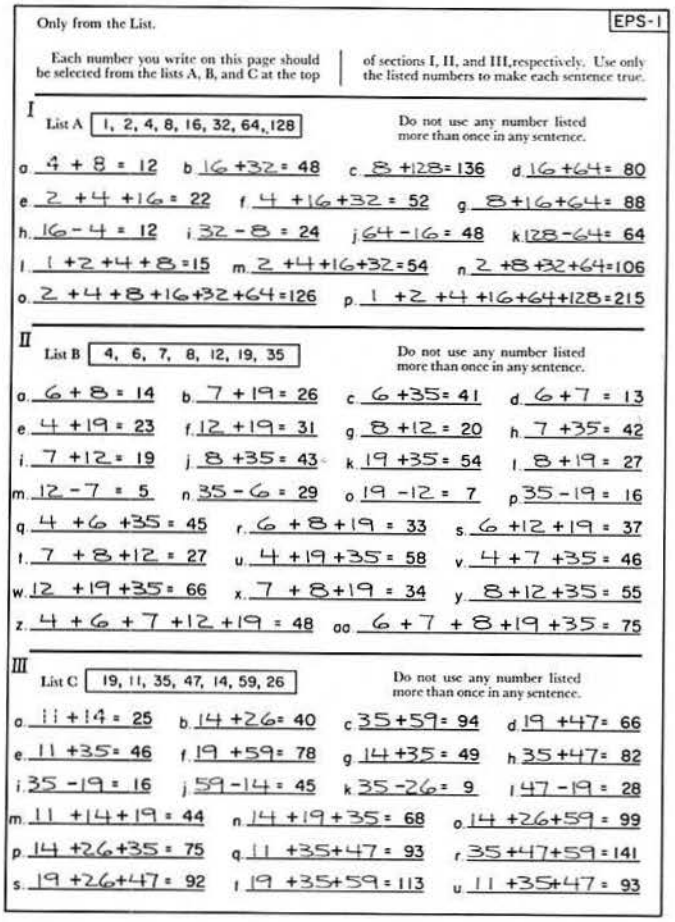
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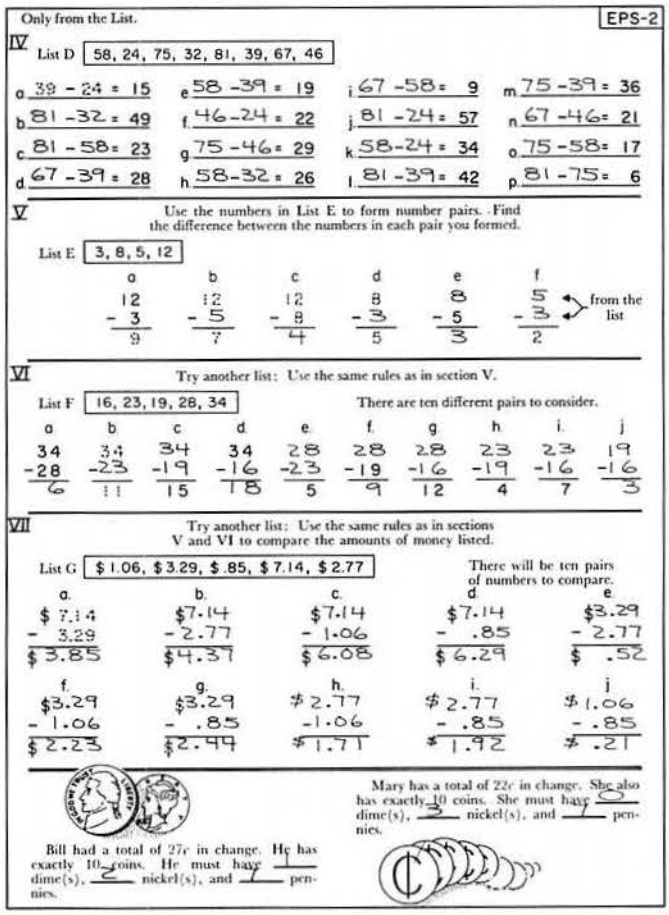
$$\underline{\quad} + \underline{\quad} + \underline{\quad} = 48$$

Practice sheet 2 builds on the *Only from the List* idea by asking for differences as follows:

From list D find two numbers whose difference is 19:

$$\underline{\quad} - \underline{\quad} = 19$$

Pupils have the further task on this page of comparing all pairs in lists E, F, and G. Finding all pairs requires more than casual consideration.



Only from the List. EPS-2

IV List D: 58, 24, 75, 32, 81, 39, 67, 46

a. 39 - 24 = 15 b. 81 - 32 = 49 c. 81 - 58 = 23 d. 67 - 39 = 28
e. 58 - 39 = 19 f. 46 - 24 = 22 g. 75 - 46 = 29 h. 58 - 32 = 26
i. 67 - 58 = 9 j. 81 - 24 = 57 k. 58 - 24 = 34 l. 81 - 39 = 42
m. 75 - 39 = 36 n. 67 - 46 = 21 o. 75 - 58 = 17 p. 81 - 75 = 6

V Use the numbers in List E to form number pairs. Find the difference between the numbers in each pair you formed.

List E: 3, 8, 5, 12

a	b	c	d	e	f
12	12	12	8	8	5
- 3	- 5	- 8	- 3	- 5	- 3
9	7	4	5	3	2

from the list

VI Try another list: Use the same rules as in section V.

List F: 16, 23, 19, 28, 34 — There are ten different pairs to consider.

a	b	c	d	e	f	g	h	i	j
34	34	34	34	28	28	28	23	23	19
-28	-23	-19	-16	-23	-19	-16	-19	-16	-16
6	11	15	18	5	9	12	4	7	3

VII Try another list: Use the same rules as in sections V and VI to compare the amounts of money listed.

List G: $1.06, $3.29, $.85, $7.14, $2.77 — There will be ten pairs of numbers to compare.

a.	b.	c.	d.	e.
$7.14	$7.14	$7.14	$7.14	$3.29
- 3.29	- 2.77	- 1.06	- .85	- 2.77
$3.85	$4.37	$6.08	$6.29	$ .52

f.	g.	h.	i.	j
$3.29	$3.29	$2.77	$2.77	$1.06
- 1.06	- .85	-1.06	- .85	- .85
$2.23	$2.44	$1.71	$1.92	$ .21

Bill had a total of 27¢ in change. He has exactly 10 coins. He must have 1 dime(s), 2 nickel(s), and 7 pennies.

Mary has a total of 22¢ in change. She also has exactly 10 coins. She must have 2 dime(s), 3 nickel(s), and 5 pennies.

THE SCIENCE FAIR PROJECT OF ALEC MARSON

It was truly an odd looking machine that Alec Marson entered in the Fairview School Science Fair. When the principal, Mr. Wilson, looked for Alec, he was gone. He had left town with his family.

"No one knows where they have gone," Mr. Wilson explained to the judges of the Science Fair. "In fact, no one knows where they came from."

The judges were puzzled by Alec's machine.

The machine had no covers and no screws to take out. It seemed as if it were made out of a single piece of strange metal.

What is it?

What does it do?

Should it be opened up to see what's inside?

An Experiment

"Let's see if anything happens when I write on the green screen that's marked, 'You write here,' " said one of the judges. "I suppose I could use that thing that looks like a pencil wired into the machine."

He wrote a 2 on the green screen. The other pencil wrote a 4 on the blue screen.

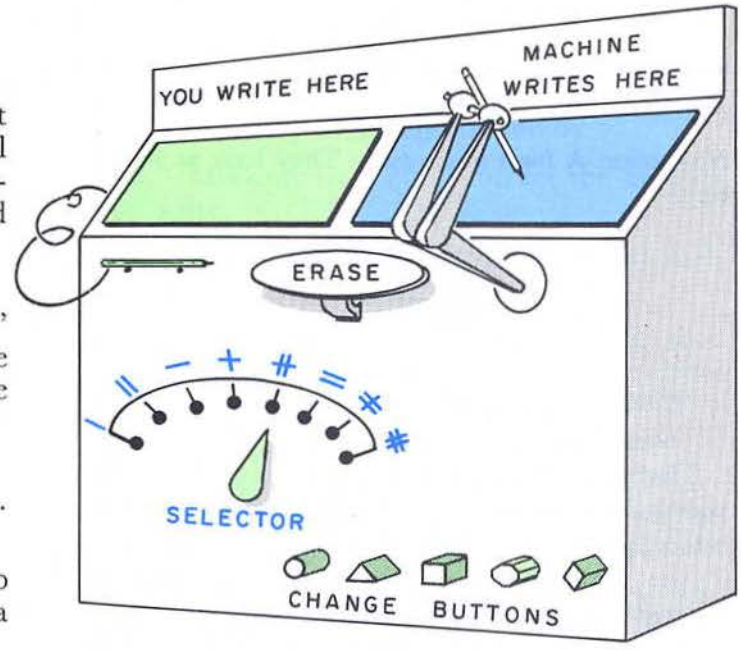
"Something happened; but I'm not sure I know anything more than I knew before."

"Let me try it," another judge asked. He pushed the ERASE key. The green and blue screens went blank. He wrote 0 on the green screen. The machine wrote 0 on the blue screen.

Another judge pushed the ERASE key, and then wrote 5 on the green screen.

The machine wrote 10 on the blue screen.

"That's interesting," Mr. Wilson said. "Let's keep a record of these experiments."



He drew a table on the board to keep track of the results. Here is a part of the table. Please fill in the missing numbers.

You write	2	0	5	1	7	25	9
Machine writes	4	0	10	2	14	50	18

Judge Harris said, "I think the machine adds 5 to any number written; then doubles the result; and, finally, subtracts 10."

(*Please fill in the missing numbers in the table.*)

Machine does:

You write	add 5	double	subtract 10	Machine writes
2	7	14	4	4
0	5	10	0	0
5	10	20	10	10
1	6	12	2	2
7	12	24	14	14
25	30	60	50	50
9	14	28	18	18
23	28	56	46	46
11	16	32	22	22

3-4 Using Alec's machine to explore basic operations of addition, subtraction, multiplication, and division

Tasks: *Pupils study the responses of Alec Marson's machine to a certain input of numbers, and they try to determine a rule of formation. They then complete two tables which are records of inputs, operations on inputs, and responses.*

Purpose: To provide experience in exploring structure and in discovering patterns. To provide meaningful and interesting practice in computation.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Everyone is sure he knows that Alec's machine is simply doubling the number written on the screen. If you assume that, all your predictions will come true. But are you sure that this is the procedure the machine follows?

Judge Harris isn't sure. He says that the machine adds 5, doubles the result, and subtracts 10. Judge Thornton has another theory. He says that it multiplies by 6 and divides by 3. There are, of course, many procedures the machine might be following.

Urge pupils to make up their own theories about ways the machine might get the same results. Some may be very obvious, such as: *Double the number, add 2, and then subtract 2;* or: *Double, subtract 3, and add 3.*

Play a game based on this idea. Two teams can play, team A and team B. First, team A writes a number. Team B responds with a second number according to some secret rule. This continues until team A thinks it knows team B's rule. Team A then says, "If we write ___ (some number), team B will respond with ___."

If team A is correct, that round is completed, and team B gets one point for every number A wrote down. The teams then switch roles. If team A is wrong, team B gives the correct answer and gets five points as well as one for each number team A wrote. Team B then selects another rule and the round continues.

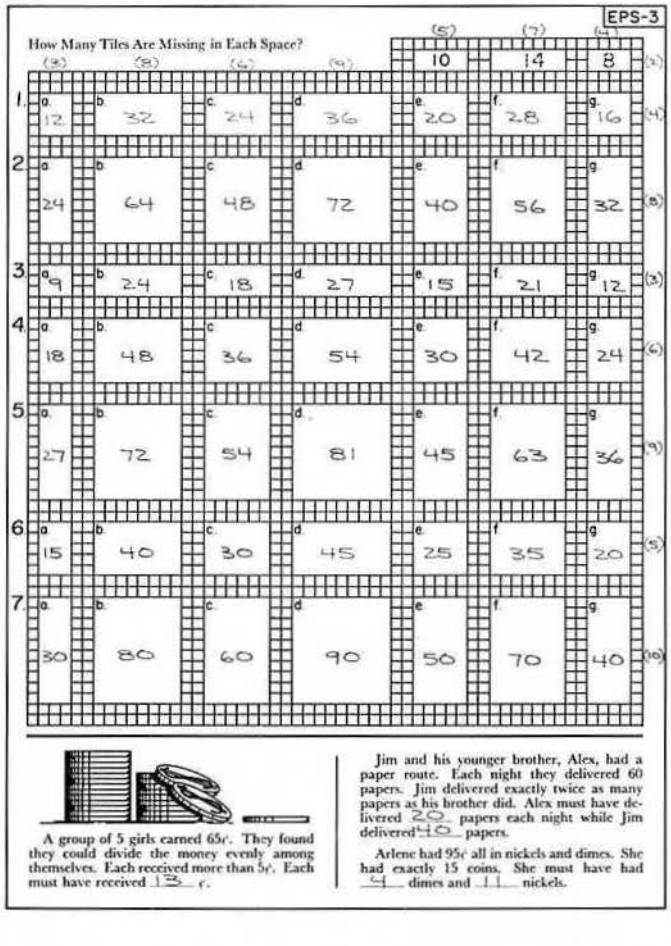
For example: Suppose that team B decides on the rule *Double and subtract 4.* When A calls a 4, B answers with a 4, because $2 \times 4 = 8$ and $8 - 4 = 4$.

When A calls out 5, B answers 6, because $2 \times 5 = 10$ and $10 - 4 = 6$. Here is a record of the round so far:

	1	2	3	4	5	
A writes	4	5	7			
B writes	4	6	10			

Now team A has a decision to make. It may see its numbers as beginning a sequence 4, 5, 6, 7 and the numbers B uses as answers as a sequence 4, 6, 8, 10. So, if they call out a 6, B will answer with an 8. Are they sure? If they are wrong, B will get five points and the round will continue. Guessing can be expensive.

They decide to risk another point rather than guess. They call out a 7, and B answers with a 10.



Now team A feels quite sure. They look at it this way:

	4	5	6	7
	4	6		10

"If we call 6, you will answer 8."

"That's right," team B answers.

That's the end of the round, and B collects three points—because team A had to test three numbers before it discovered B's rule.

Now A makes up a rule, and B calls out numbers. A decides on the rule *Subtract 2 and triple the result.* B calls out a 5, and A answers with a 9, because $5-2=3$ and $3\times3=9$. B calls out 1, and A is in trouble unless A's members are familiar with negative numbers and know what 3×-1 means. If not, A simply replies, "Your number is too small; pick another." And the score remains unchanged.

B calls out 2, and A answers 0, because $2-2=0$ and $3\times0=0$. The record of the round looks like this:

	1	2	3	4		
B	5	2				
A	9	0				

That's not much help. So team B works out a plan of action. They will fill in one more row in a chart and see if this reveals a pattern:

2 → 0

3 → 3

4 → ---

5 → 9

So they decide to call out a 3, and A answers with a 3, which B sees as another item in its chart. Now B has to decide what A will answer if B calls out 4. The pattern seems to suggest a 6 to fill out the 0, 3, ___, 9 sequence.

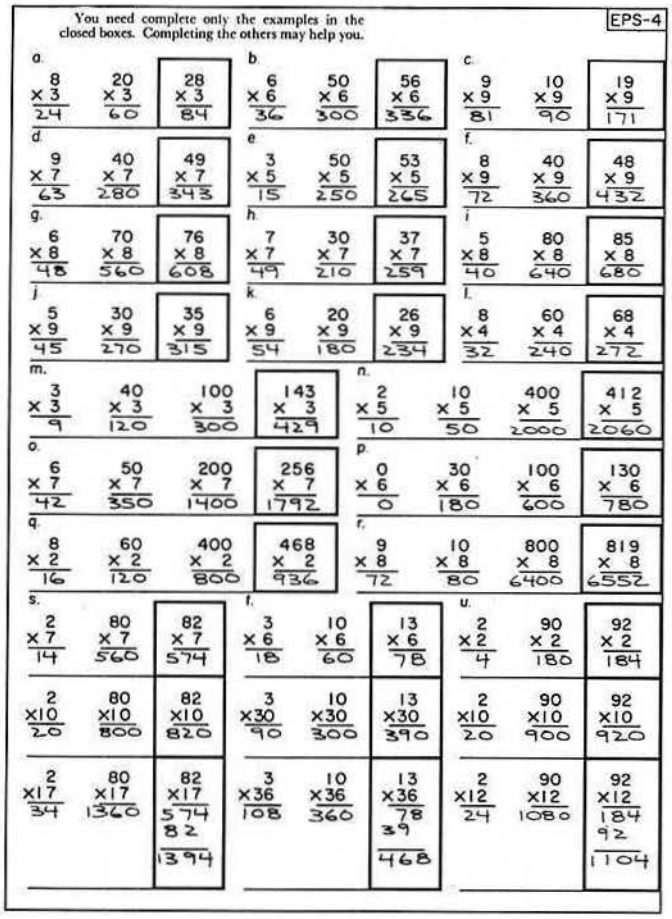
"If we call 4, you will answer 6."

"That's right."

So B gets three points. And another round begins.

Practice sheet 3 ought to present no difficulty at all if the pupils understand the idea of multiplication and know the basic combinations.

Practice sheet 4 deals with a multiplication algorithm. Note that the examples are clustered. See, especially, exercises *s*, *t*, and *u*.



ANOTHER SUGGESTION

Judge Thornton said, "Perhaps the machine multiplies the number you have written by 6, and then divides the result by 3."

(*Please fill in the missing numbers in the table.*)

You write	Machine's Work		Machine writes
	multiply by 6	divide by 3	
2	12	4	4
0	0	0	0
5	30	10	10
1	6	2	2
7	42	14	14
25	150	50	50

Mr. Wilson, the principal, suggested that perhaps the machine wasn't as complicated as the judges thought it was.

"Maybe it simply doubles whatever number you write."

Judge Jones offered another explanation.

"Maybe the machine adds 3, triples that sum, subtracts 8 from the product, and finally subtracts 1 more than the number you write."

You write	add 3	triple	subtract 8	subtract 1 more than the no. you write	Mach. writes
2	5	15	7	4	4
0	3	9	1	0	0
5	8	24	16	10	10
1	4	12	4	2	2
7	10	30	22	14	14
25	28	84	76	50	50
9	12	36	28	18	18
23	26	78	70	46	46
11	14	42	34	22	22
10	13	39	31	20	20
17	20	60	52	34	34
50	53	159	151	100	100

MORE SUGGESTIONS

Other suggestions were as follows:

a. Multiply by 10 and divide by 5.

b. Multiply by 3 and subtract the number you write.

c. Halve the number you write and multiply the result by 4.

d. Subtract 5, double the result, and then add 10. (Do you think the machine can subtract 5 from 0 or from 1 or 2 or 3 or 4?)

Mr. Wilson was a little impatient. "I'm sure," he said, "that Alec Marson could have built a machine that got the same results by an even more complicated sequence of operations.

"He might have had his machine add 2 to the number you write; multiply by 6, add 9, divide by 3, and subtract 7."

(*Please fill in the missing numbers in the table.*)

You write	2	0	5	1	7	3	4
add 2	4	2	7	3	9	5	6
multiply by 6	24	12	42	18	54	30	36
add 9	33	21	51	27	63	39	45
divide by 3	11	7	17	9	21	13	15
subtract 7	4	0	10	2	14	6	8
Machine writes	4	0	10	2	14	6	8

"However," Mr. Wilson continued, "Alec Marson was a very bright boy. He seldom used complicated methods to do something if he could find a way to do the same thing more simply.

"I'm convinced he built a machine that simply doubles any number you write on the green screen.

"I think that there is a lot more to Alec's machine than we think we know about yet. Let's see what else it can do."

Tasks: *Pupils study the responses of Alec Marson's machine to a certain input of numbers, and they try to determine a rule of formation. They then complete two tables which are records of inputs, operations on inputs, and responses.*

Purpose: To provide experience in exploring structure and in discovering patterns. To provide meaningful and interesting practice in computation.

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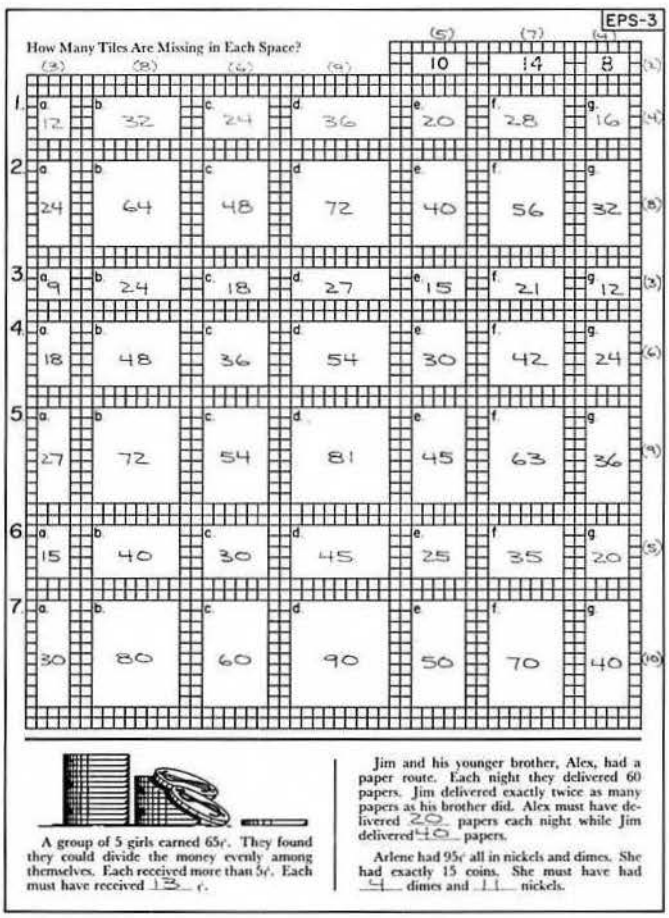
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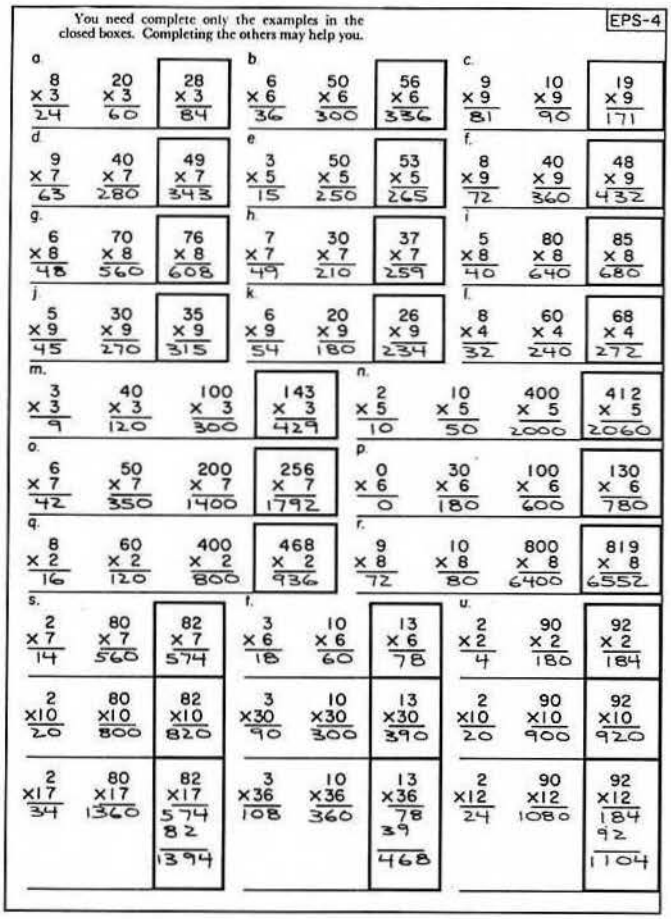
"If we call 4, you will answer 6."

"That's right."

So B gets three points. And another round begins.

Practice sheet 3 ought to present no difficulty at all if the pupils understand the idea of multiplication and know the basic combinations.

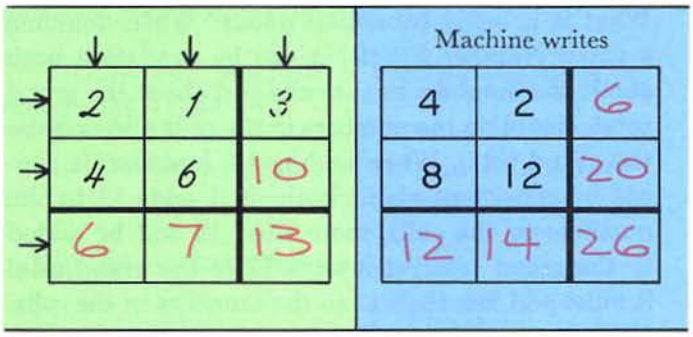
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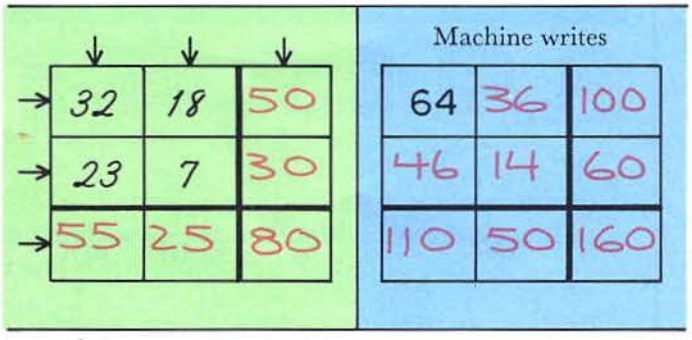
Mr. Wilson explained that Alec had thought Cross Number Puzzles were fun. The judges had never heard of such puzzles.

"In one form," Mr. Wilson explained, "four numbers are written in a group of four boxes.

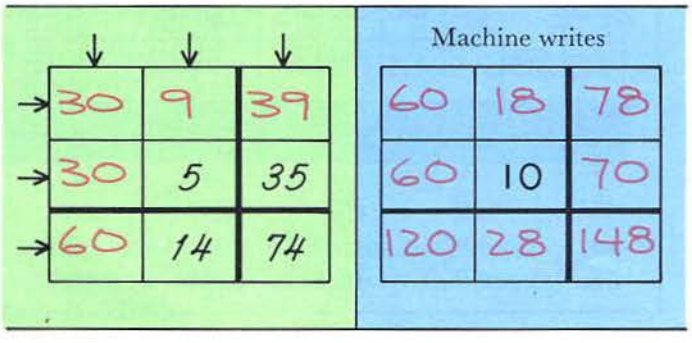
You add the numbers in each row and each column separately, and then add each pair of sums." Mr. Wilson illustrated by drawing on the green screen. To everyone's amazement, the machine started writing on the blue screen.



"Amazing!" the judges said in unison. Judge Thornton wanted to try a Cross Number Puzzle with larger numbers.



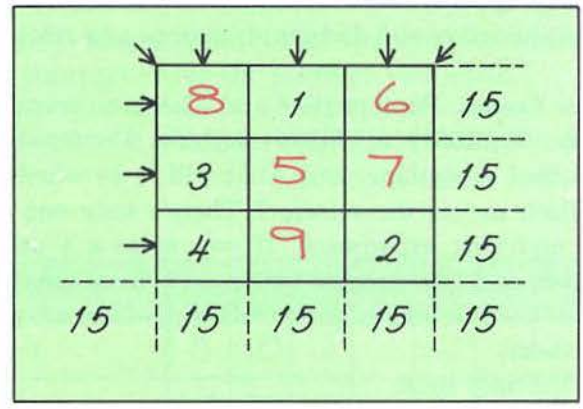
Mr. Wilson said, "Alec knew how to complete puzzles when the numbers were arranged differently. Perhaps the machine also knows how to complete such puzzles."



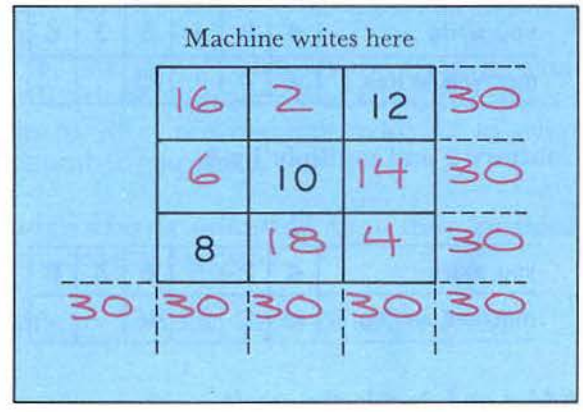
Judge Jones shook his head. "That boy certainly is unusual."

"Yes," replied Mr. Wilson, "he is, indeed. He also enjoyed working with Magic Squares. I wonder what would happen if I put a Magic Square on the green screen."

Mr. Wilson started a Magic Square. The machine completed it for him. Please complete this Magic Square.



The machine also wrote on the blue screen. Please finish this Magic Square.



Judge Thornton said, "I agree with Mr. Wilson. The machine is set to double everything. Let's call it a Doubling Machine."

"Not yet," Mr. Wilson cautioned. "I want to know what happens when we move the pointer along the dial marked Selector."

5 Using Alec's machine to construct Cross Number Puzzles and Magic Squares

Tasks: *Pupils study the responses of Alec Marson's machine to a certain input of* Cross Number Puzzles *and* Magic Squares, *and they try to determine a rule of formation. They then complete all* Cross Number Puzzles *and* Magic Squares.

Purpose: To provide experience in exploring structure and in discovering patterns. To provide meaningful and interesting practice in computation.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: On pupil page 5 the *Cross Number Puzzle* is introduced for the first time in Level E. Along with its cousin, the *Magic Square*, this fascinating device will be employed often in the next nine pages. Mr. Wilson explains the rules for the simplest *Cross Number Puzzle:*

> Add the numbers listed in the top row, and write the sum at the right. Do the same with the second row. Then add the numbers listed in the left-hand column, and write the sum at the bottom. Do the same with the second column.

Now, if the puzzle "works," you can write a number in the check box (lower right-hand corner, in this case) that is the sum of both the two numbers listed above it and the two on the left (as in the first example: $13=10+3$ and $13=6+7$).

But as soon as Mr. Wilson put such a puzzle on one screen on the machine, the machine put another on the other screen. What is the relation between the two puzzles? Instead of trying to use words to explain the pattern, the judges try another example—and test their ability to predict results.

In the third pair of puzzles at the bottom of the left half of pupil page 5, the puzzle requires some "reverse" arguments. If 74 is written in the check box and 35 is the sum of the numbers in the second row then the sum of the top row must be 39, because 39 is the only number whose sum with 35 is 74. Similarly, 60 must be the sum of the numbers in the left-hand column.

Further, if 35 is the sum of the second row and 5 is one of the addends then the other must be 30.

With such arguments the puzzle can be completely reconstructed. Then by knowing the relationship between the puzzle of the judges and the puzzle with which the machine responds we can predict the machine's next move.

When Mr. Wilson tries a Magic Square, the machine adopts the same response pattern.

"Is the response of the machine also a Magic Square?" (Yes.)

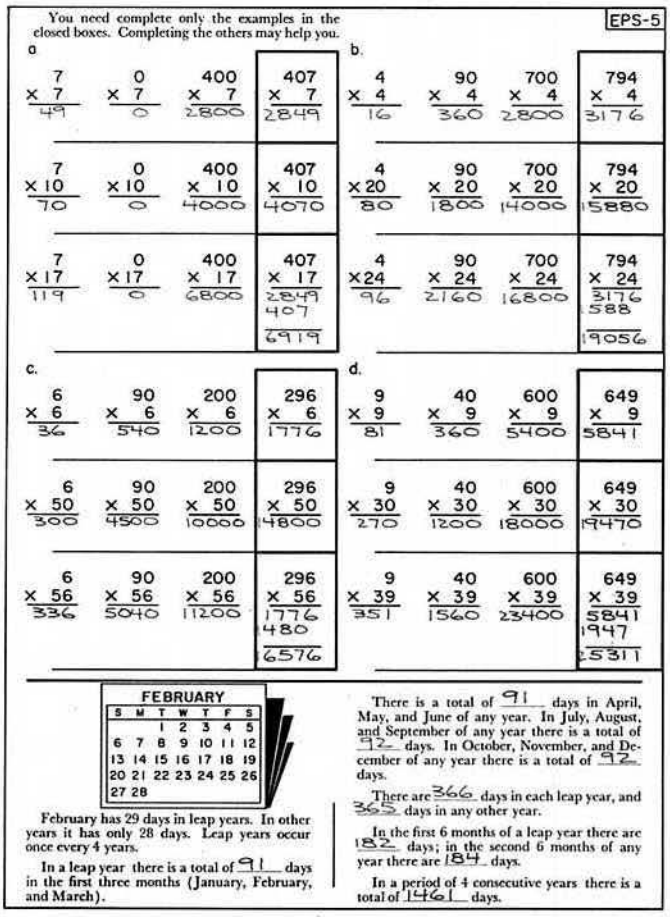
Here are other Magic Squares you might try on the machine:

18	21	6
3	15	27
24	9	12

7	17	18
25	14	3
10	11	21

46	15	41
29	34	39
27	53	22

Practice sheet 5 deals with a multiplication algorithm. If pupils can handle the examples in the four closed boxes directly, they ought to be excused from doing the other examples which are merely decompositions of the boxed examples.


You need complete only the examples in the closed boxes. Completing the others may help you.

EPS-5

a.

7 × 7 = 49	0 × 7 = 0	400 × 7 = 2800	407 × 7 = 2849
7 × 10 = 70	0 × 10 = 0	400 × 10 = 4000	407 × 10 = 4070
7 × 17 = 119	0 × 17 = 0	400 × 17 = 6800	407 × 17 = 2849 + 407 = 6919

b.

4 × 4 = 16	90 × 4 = 360	700 × 4 = 2800	794 × 4 = 3176
4 × 20 = 80	90 × 20 = 1800	700 × 20 = 14000	794 × 20 = 15880
4 × 24 = 96	90 × 24 = 2160	700 × 24 = 16800	794 × 24 = 3176 + 588 = 19056

c.

6 × 6 = 36	90 × 6 = 540	200 × 6 = 1200	296 × 6 = 1776
6 × 50 = 300	90 × 50 = 4500	200 × 50 = 10000	296 × 50 = 14800
6 × 56 = 336	90 × 56 = 5040	200 × 56 = 11200	296 × 56 = 1776 + 4800 = 6576

d.

9 × 9 = 81	40 × 9 = 360	600 × 9 = 5400	649 × 9 = 5841
9 × 30 = 270	40 × 30 = 1200	600 × 30 = 18000	649 × 30 = 19470
9 × 39 = 351	40 × 39 = 1560	600 × 39 = 23400	649 × 39 = 5841 + 1947 = 25311

FEBRUARY

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28					

February has 29 days in leap years. In other years it has only 28 days. Leap years occur once every 4 years.

In a leap year there is a total of 91 days in the first three months (January, February, and March).

There is a total of 91 days in April, May, and June of any year. In July, August, and September of any year there is a total of 92 days. In October, November, and December of any year there is a total of 92 days.

There are 366 days in each leap year, and 365 days in any other year.

In the first 6 months of a leap year there are 182 days; in the second 6 months of any year there are 184 days.

In a period of 4 consecutive years there is a total of 1461 days.



Mr. Wilson turned the pointer from # to #. He wrote 4 on the green screen. The machine wrote 16 on the blue screen.

"I think the machine is multiplying by 4," said Judge Thornton.

Mr. Wilson wrote 6 and the machine wrote 18.

Did the machine make a mistake?

Mr. Wilson made a table. Please fill in the missing numbers in the table.

A.

You write	4	6	12	9	13	24	25
Machine writes	16	18	24	21	25	36	37

Mr. Wilson concluded that the machine did not make a mistake.

Mr. Thornton said, "The machine adds 6 to the number written, doubles the result and subtracts the number you write.

"Please fill in the missing numbers in my table."

B.

You write	4	6	12	9	13	24	25
Add 6.	10	12	18	15	19	30	31
Double the sum.	20	24	36	30	38	60	62
Subtract the no. you write.	16	18	24	21	25	36	37
Machine writes	16	18	24	21	25	36	37

Judge Jones said, "The machine doubles the number you write, adds 24 to the result, and then divides by 2.

"Please fill in the missing numbers in my table."

C.

You write	4	6	12	9	13	24	25
Double the no.	8	12	24	18	26	48	50
Add 24.	32	36	48	42	50	72	74
Divide by 2.	16	18	24	21	25	36	37
Machine writes	16	18	24	21	25	36	37

Judge Harris said, "Add 4, triple the result, and subtract twice the number you write."

"Please fill in the missing numbers in my table."

D.

You write	4	6	12	9	13	24	25
Add 4.	8	10	16	13	17	28	29
Triple the sum.	24	30	48	39	51	84	87
Subtract twice no. you write.	24 - 8	30 - 12	48 - 24	39 - 18	51 - 26	84 - 48	87 - 50
Machine writes	16	18	24	21	25	36	37

Mr. Wilson was growing impatient again.

"No," he said. "Alec always tried to avoid complications. I'm sure that when the selector points to # , the machine adds 12 to whatever number you write.

Judge Harris admitted that they certainly had missed the easy way.

Let's try another Cross Number Puzzle.

E.

You write here.		
9	7	16
20	30	50
29	37	66

Machine writes here.		
12	10	22
23	33	56
35	43	78

6-7 Using Alec's machine to further explore the basic arithmetic operations

Tasks: *Pupils study the responses of Alec Marson's machine to the input of certain numbers and to a* Cross Number Puzzle *and determine the rules of formation. They then complete the tables and* Cross Number Puzzle.

Purpose: To provide experience in exploring structure and in discovering patterns. To provide meaningful and interesting practice in computation.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Pupil pages 6 and 7 begin to reveal the great flexibility of Alec's machine. Previously, it doubled everything. But what will it do when Mr. Wilson moves the selector? There's only one way to find out: experiment. If you write a 4 on the screen and the machine writes a 16, there are many rules the machine might be following. Here are some of them:

Multiply by 4.

you write	4	2	5	9	12
machine writes	16	8	20		

Multiply the number by itself.

you write	4	2	1	5	3	6
machine writes	16	4	1	25		

Subtract 2 and multiply by 8.

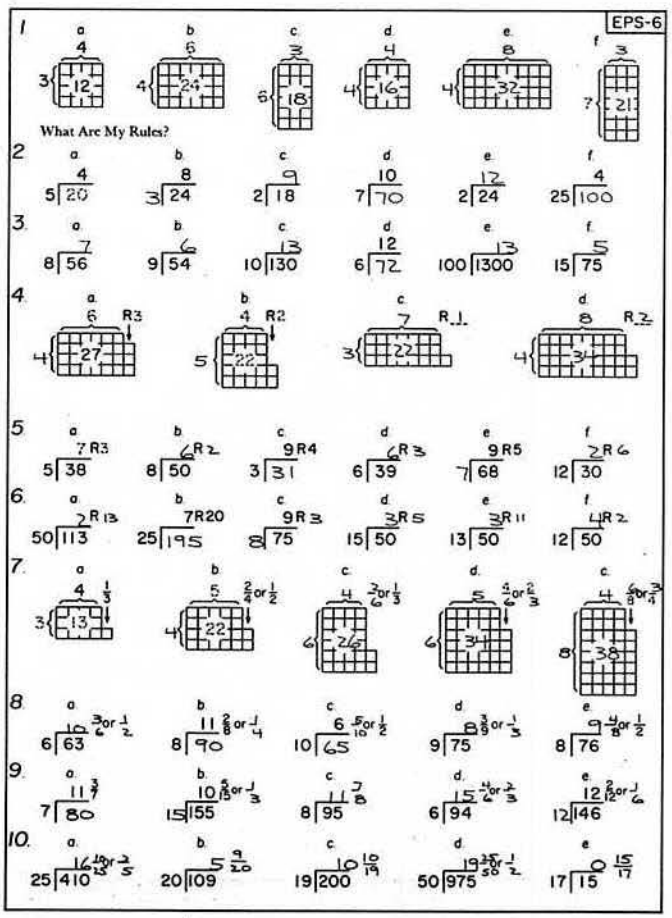
you write	4	2	5	8	3	6
machine writes	16	0	24	48		

Add 4 and double the result.

you write	4	0	5	2	7
machine writes	16	8	18		

These rules are not equivalent, even though they all assign 16 to 4. Have pupils find some rules that are equivalent. For example, the rules *add 12* and *add 20 and subtract 8 from the result* are equivalent—they produce the same results. Why? Could it be because $12 = 20 - 8$? This study will help pupils understand why some of the complicated rules the judges dream up produce the same results.

When Alec's machine comes to adding 12 to a *Cross Number Puzzle*, it may seem at first to be behaving inconsistently. In fact, the problem is: What is it being consistent *about*? When doubling a *Cross Number Puzzle*, it can be consistent both about the numbers in the cells and about the grand total—doubling the numbers in the cells also doubles the grand total. When adding 12, however, it cannot be consistent about both—if it adds 12 to the numbers in the cells, more than 12 will be added to the grand total; if it adds 12 to the grand total it must add less than 12 to the numbers in the cells. Since it chooses to add 3 to each of the cells, it is evident that Alec's machine has been constructed so as to be consistent about the grand total of *Cross Number Puzzles* rather than about the numbers in the cells.



Experimenting with the Selector

The judges moved the selector to each of the eight positions indicated.

Mr. Wilson kept track of the results, some of which are shown in the tables below.

Please fill in the missing numbers in each table.

Selector position ↓

		a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
|	You write	1	2	5	14	10	9	0	8	50	75	106	500	610
	Machine writes	3	5	11	29	21	19	1	17	101	151	213	1001	1221
||	You write	0	1	2	3	5	7	50	6	44	72	110	314	350
	Machine writes	3	6	9	12	18	24	153	21	135	219	333	945	1053
−	You write	10	2	7	1	11	3	8	5	15	25	31	200	320
	Machine writes	19	3	13	1	21	5	15	9	29	49	61	399	639
+	You write	3	1	5	2	4	7	6	9	8	20	25	307	515
	Machine writes	18	6	30	12	24	42	36	54	48	120	150	1842	3090
#	You write	0	7	1	4	19	36	49	28	100	125	351	161	199
	Machine writes	0	14	2	8	38	72	98	56	200	250	702	322	398
=	You write	2	1	5	10	4	6	9	3	8	100	111	220	213
	Machine writes	9	5	21	41	17	25	37	13	33	401	445	881	853
≠	You write	0	11	15	23	13	21	12	14	25	18	22	92	124
	Machine writes	0	44	60	92	52	84	48	56	100	72	88	368	496
#	You write	1	7	8	48	0	29	50	58	88	108	318	490	899
	Machine writes	13	19	20	60	12	41	62	70	100	120	330	502	911

When the pointer of Alec Marson's machine was at " ≠ ," Mr. Wilson tried another Cross Number Puzzle. He used some numbers which were larger than those he used in the previous puzzles.

Please complete the puzzles.

You write here.

	add	add	add	add
add →	100	20	7	127
add →	300	70	9	379
add →	400	90	16	506

The machine writes here.

400	80	28	508
1200	280	36	1516
1600	360	64	2024

Tasks: *Pupils study the responses of Alec Marson's machine to the input of certain numbers and to a* Cross Number Puzzle *and determine the rules of formation. Then they complete the tables and* Cross Number Puzzle.

Purpose: To provide experience in exploring structure and in discovering patterns. To provide meaningful and interesting practice in computation.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: As the selector is moved from point to point, the machine changes its rule. With each new setting, there's a new game to play. The first rule on pupil page 7 might be *multiply by 2 and add 1.*

You might explain that you have found a way to help uncover many of the machine's rules. It usually multiplies, or multiplies and then adds or subtracts. So you make a chart:

you write	1	2	5	14
machine writes	3	5	11	29
2x what you write	2	4	10	28
3x what you write	3	6	15	42
4x what you write	4	8	20	64

You notice that when you multiply by 2 you get a number that is 1 less than the number the machine writes. So you know that one rule might be *multiply by 2 and add 1 to the result.*

Try this approach to the second set of responses on pupil page 7.

you write	0	1	2	3	5
machine writes	3	6	9	12	18
2x what you write	0	2	4	6	10
3x what you write	0	3	6	9	15
4x what you write	0	4	8	12	20

The machine increases by 3 each number in your "3 ×" line. So one rule the machine might be following is *multiply by 3 and add 3.* Of course, you aren't sure. The machine might follow a rule such as *add 1 and multiply by 3,* which is equivalent to *multiply by 3 and add 3.*

This approach will help in every example except the last, which might be simply *add 12* or *multiply by 1 and add 12.* Rules for the other examples might be:

— "multiply by 2 and then add 1"
+ "multiply by 6"
"multiply by 2"
= "multiply by 4 and add 1"
≠ "multiply by 4"
"add 12"

Remember, however, that there are many rules which are equivalent to these.

Practice sheets 7 and 8 are provided to find out how familiar pupils are with the basic ideas of fractions as used in the context of familiar systems of measure.

EPS-7

A. 1 dozen = 12 eggs	B. 1 hour = 60 minutes	C. 1 yard = 36 inches
1 $\frac{1}{2}$ doz = 6 eggs	1 $\frac{1}{2}$ hr = 30 min	1 $\frac{1}{2}$ yd = 18 in
2 $\frac{1}{3}$ doz = 4 eggs	2 $\frac{1}{3}$ hr = 20 min	2 $\frac{1}{3}$ yd = 12 in
3 $\frac{1}{4}$ doz = 3 eggs	3 $\frac{1}{4}$ hr = 15 min	3 $\frac{1}{4}$ yd = 9 in
4 $\frac{1}{12}$ doz = 1 eggs	4 $\frac{1}{12}$ hr = 5 min	4 $\frac{1}{12}$ yd = 3 in
5 $\frac{2}{3}$ doz = 8 eggs	5 $\frac{2}{3}$ hr = 40 min	5 $\frac{2}{3}$ yd = 24 in
6 $\frac{3}{4}$ doz = 9 eggs	6 $\frac{3}{4}$ hr = 45 min	6 $\frac{3}{4}$ yd = 27 in
7 $\frac{5}{12}$ doz = 5 eggs	7 $\frac{5}{12}$ hr = 25 min	7 $\frac{5}{12}$ yd = 15 in
8 $\frac{1}{6}$ doz = 2 eggs	8 $\frac{1}{6}$ hr = 10 min	8 $\frac{1}{6}$ yd = 6 in
9 $\frac{5}{6}$ doz = 10 eggs	9 $\frac{5}{6}$ hr = 50 min	9 $\frac{5}{6}$ yd = 30 in
10 $1\frac{1}{2}$ doz = 18 eggs	10 $1\frac{1}{2}$ hr = 90 min	10 $1\frac{1}{2}$ yd = 54 in
11 $1\frac{1}{4}$ doz = 15 eggs	11 $1\frac{1}{4}$ hr = 75 min	11 $1\frac{1}{4}$ yd = 45 in
12 $2\frac{1}{3}$ doz = 28 eggs	12 $2\frac{1}{3}$ hr = 140 min	12 $2\frac{1}{3}$ yd = 84 in
13 $1\frac{2}{3}$ doz = 20 eggs	13 $1\frac{2}{3}$ hr = 100 min	13 $1\frac{2}{3}$ yd = 60 in
14 $1\frac{5}{6}$ doz = 22 eggs	14 $1\frac{5}{6}$ hr = 110 min	14 $1\frac{5}{6}$ yd = 66 in

D.	G.	J.
$\frac{1}{2}$ doz, or 6 eggs	$\frac{1}{3}$ hr, or 20 min	$\frac{1}{12}$ yd, or 3 in
+ $\frac{1}{4}$ doz, or 3 eggs	+ $\frac{2}{3}$ hr, or 40 min	+ $\frac{5}{12}$ yd, or 15 in
$\frac{3}{4}$ doz, or 9 eggs	1 hr, or 60 min	$\frac{1}{2}$ yd, or 18 in
E.	**H.**	**K.**
$\frac{3}{4}$ doz, or 9 eggs	$\frac{1}{2}$ hr, or 30 min	$1\frac{1}{3}$ yd, or 48 in
+ $\frac{1}{2}$ doz, or 6 eggs	+ $\frac{2}{3}$ hr, or 40 min	+ $\frac{3}{4}$ yd, or 27 in
$1\frac{1}{4}$ doz, or 15 eggs	$1\frac{1}{2}$ hr, or 70 min	$2\frac{1}{12}$ yd, or 75 in
F.	**I.**	**L.**
$\frac{2}{3}$ doz, or 8 eggs	$\frac{1}{3}$ hr, or 20 min	$1\frac{1}{6}$ yd, or 42 in
− $\frac{5}{12}$ doz, or 5 eggs	− $\frac{1}{4}$ hr, or 15 min	− $\frac{2}{3}$ yd, or 24 in
$\frac{1}{4}$ doz, or 3 eggs	$\frac{1}{12}$ hr, or 5 min	$\frac{1}{2}$ yd, or 18 in

The judges decided to adjourn until the next morning.

Mr. Wilson took Alec's machine home with him.

In his study at home, he pushed the triangular button. Two Cross Number Puzzle forms flashed on the green screen and one flashed on the blue screen. Mr. Wilson went to work. Finish Mr. Wilson's work for him.

A.

You write here.

20	8	28
30	4	34
50	12	62

+

9	10	19
7	6	13
16	16	32

Machine writes here.

29	18	47
37	10	47
66	28	94

Here are some more puzzles started by Mr. Wilson.

The green forms appeared on the green screen. The blue forms on the blue screen. Please complete the puzzles.

B.

51	13	64
20	16	36
71	29	100

+

50	8	58
30	7	37
80	15	95

=

101	21	122
50	23	73
151	44	195

C.

11	30	41
10	7	17
21	37	58

+

8	25	33
15	11	26
23	36	59

=

19	55	74
25	18	43
44	73	117

D.

100	24	124
200	16	216
300	40	340

+

300	51	351
100	58	158
400	109	509

=

400	75	475
300	74	374
700	149	849

E.

45	350	395
32	110	142
77	460	537

+

29	125	154
28	30	58
57	155	212

=

74	475	549
60	140	200
134	615	749

8 Using Alec's machine to add Cross Number Puzzles

Tasks: *Pupils study the responses of Alec Marson's machine to the input of* Cross Number Puzzles *and determine the rules of formation. They then complete all* Cross Number Puzzles.

Purpose: To provide experience in exploring structure and in discovering patterns. To provide meaningful and interesting practice in computation.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

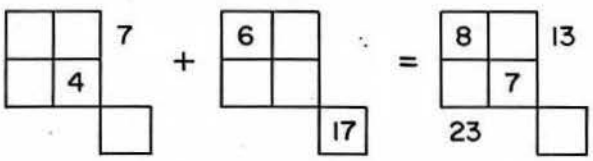
The Lesson: Here Alec's machine seems to be "adding" *Cross Number Puzzles.* This notion may or may not be new to the students. MATH WORKSHOP has explored similar ideas before. Throughout arithmetic thus far, from its very beginning in kindergarten, pupils have thought of adding things to things—and getting things as the result:

3 cows + 2 cows = 5 cows
$7.24 + $9.35 = $16.59
a quarter + a dime = 35¢

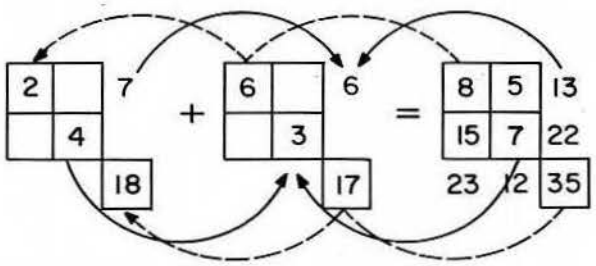
Here we add a *Cross Number Puzzle* to another in such a way that the result is a *Cross Number Puzzle.*

Each *Cross Number Puzzle* given here involves the task of reconstruction since the given information occurs in a variety of arrangements. The first two puzzles in exercise B can be reconstructed separately, without considering the resulting third puzzle, or sum. However, in exercises C, D, and E at least one puzzle can only be reconstructed by working backwards from the sum.

Here is another example requiring a maximum of such work; it may be used as board work to familiarize the class with the idea.

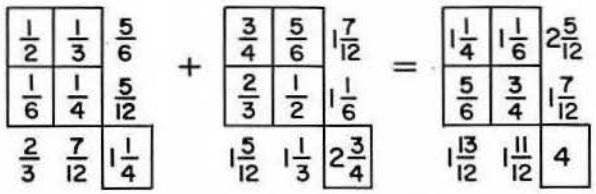


Notice that the sum-puzzle can be completed with just the given information. The arrows suggest further steps toward reconstruction.



Now the rest of the job is routine.

Some may like to try adding puzzles involving fractions:



In this way you can tailor the activity to challenge every member of the class.

Practice sheets 7 and 8 are provided to find out how familiar pupils are with the basic ideas of fractions as used in the context of familiar systems of measurement.

EPS-8

A

1	pound =	16	ounces		
1. 1/2	lb. =	8	oz.		
2. 1/8	lb. =	2	oz.		
3. 3/8	lb. =	6	oz.		
4. 1	foot =	12	inches		
5. 3/4	ft. =	9	in.		
6. 1/12	ft. =	1	in.		

B

1	week =	7	days		
1. 1/7	wk. =	1	da.		
2. 1 1/7	wk. =	8	da.		
3. 1 3/7	wk. =	10	da.		
4. 1	gallons =	4	quarts		
5. 1/4	gal. =	1	qt.		
6. 2 3/4	gal. =	11	qt.		

C

1	dime =	10	pennies		
1. 1/2	dime =	5	pen.		
2. 1/10	dime =	1	pen.		
3. 1/5	dime =	2	pen.		
4. 3/5	dime =	6	pen.		
5. 1 3/10	dimes =	13	pen.		
6. 3 2/5	dimes =	34	pen.		

D

1/4	lb., or	4	oz.		
+ 1/8	lb., or	2	oz.		
3/8	lb., or	6	oz.		

E

3/7	wk., or	3	da.		
+ 6/7	wk., or	6	da.		
1 5/7	wk., or	9	da.		

F

1/2	dime, or	5	pen.		
+ 1/5	dime, or	2	pen.		
7/10	dime, or	7	pen.		

G

1/3	ft., or	4	in.		
+ 5/12	ft., or	5	in.		
3/4	ft., or	9	in.		

H

1/2	gal., or	2	qt.		
+ 3/4	gal., or	3	qt.		
1 1/4	gal., or	5	qt.		

I

1 1/10	dimes, or	11	pen.		
+ 1 2/5	dimes, or	14	pen.		
2 1/2	dimes, or	25	pen.		

J

1 1/4	ft., or	15	in.		
- 7/12	ft., or	7	in.		
2/3	ft., or	8	in.		

K

1 1/2	lb., or	24	oz.		
- 7/8	lb., or	14	oz.		
5/8	lb., or	10	oz.		

L

1 1/2	dime, or	15	pen.		
- 4/5	dime, or	8	pen.		
7/10	dime, or	7	pen.		

Think about dozens, hours, yards, pounds, weeks, dimes, feet, and gallons to help you work the exercises below.

M

1. 1/2 + 1/4 = 3/4
2. 4/5 + 4/5 = 1 3/5
3. 7/10 + 2 1/10 = 2 4/5
4. 1/3 − 1/4 = 1/12

N

1. 1 2/7 + 2 4/7 = 3 6/7
2. 3/8 + 5/8 = 1
3. 7/16 + 1/16 = 1/2
4. 1 − 3/7 = 4/7

O

1. 1/2 + 1/10 = 3/5
2. 1/6 + 1/2 = 2/3
3. 1 3/5 + 7/10 = 2 3/10
4. 4 1/2 − 1 3/4 = 2 3/4

P

1. 1 1/4 + 3/4 = 2
2. 1/4 + 1 1/3 = 1 7/12
3. 3 3/4 + 3 3/4 = 7 1/2
4. 1 9/10 − 3/5 = 1 1/2

"I wonder what will happen if I move the pointer over to ' // ,' " said Mr. Wilson. He left the triangular button pressed down.

Two Cross Number Puzzle forms appeared on the green screen, with a minus sign between them.

A.

5	10	15
20	15	35
25	25	50

—

2	1	3
4	3	7
6	4	10

=

3	9	12
16	12	28
19	21	40

B.

22	8	30
15	39	54
37	47	84

—

3	7	10
9	12	21
12	19	31

=

19	1	20
6	27	33
25	28	53

C.

10	20	30
30	40	70
40	60	100

—

6	2	8
4	8	12
10	10	20

=

4	18	22
26	32	58
30	50	80

D.

33	30	63
20	17	37
53	47	100

—

10	12	22
11	13	24
21	25	46

=

23	18	41
9	4	13
32	22	54

E.

300	42	342
500	49	549
800	91	891

—

200	17	217
200	22	222
400	39	439

=

100	25	125
300	27	327
400	52	452

F.

150	26	176
325	80	405
475	106	581

—

90	7	97
35	36	71
125	43	168

=

60	19	79
290	44	334
350	63	413

9 Using Alec's machine to subtract Cross Number Puzzles

Tasks: *Pupils study the responses of Alec Marson's machine to the input of* Cross Number Puzzles *and determine the rules of formation. They then complete all* Cross Number Puzzles.

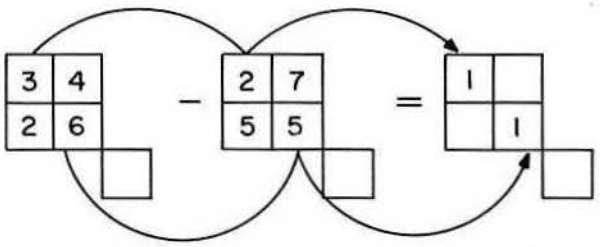
Purpose: To provide experience in exploring structure and in discovering patterns. To provide meaningful and interesting practice in computation.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: There is an important difference between adding *Cross Number Puzzles* and subtracting them, just as there is between adding numbers and subtracting them. If we are limited to whole numbers (including zero), we can express the sum for any two. The same is true of *Cross Number Puzzles*. But there are no whole numbers that can be used to complete the following sentences (as true statements):

$$5-7=\_\_\_$$
$$3-4=\_\_\_$$

Likewise, there are no whole-number *Cross Number Puzzles* that can be used to complete the following:



since there are no whole numbers we can use to complete "4−7=___" and "2−5=___." So, we see that subtraction of *Cross Number Puzzles* is possible only under certain conditions.

Still other things can be "added" and "subtracted." Consider "addition" of equations:

$$\begin{pmatrix}2\\+7\\9\end{pmatrix}+\begin{pmatrix}9\\+5\\14\end{pmatrix}=\begin{pmatrix}11\\+12\\23\end{pmatrix} \qquad \begin{pmatrix}8\\+7\\15\end{pmatrix}+\begin{pmatrix}30\\+50\\80\end{pmatrix}=\begin{pmatrix}38\\+57\\96\end{pmatrix}$$

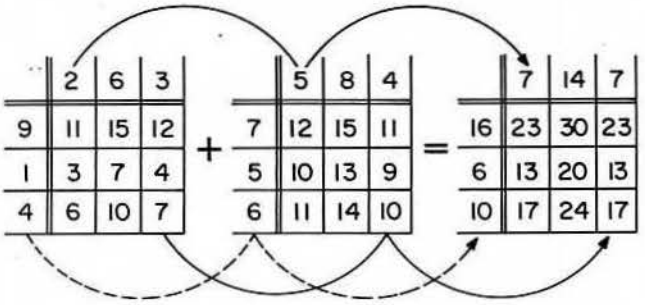
$$\begin{array}{r} 2+7=9 \\ (+)\ 9+5=14 \\ \hline 11+12=23 \end{array} \qquad \begin{array}{r} 8+7=15 \\ (+)\ 30+50=80 \\ \hline 38+57=95 \end{array}$$

Again we find that subtraction requires planning ahead if whole numbers are to be adequate.

$$\begin{pmatrix}8\\+5\\13\end{pmatrix}-\begin{pmatrix}4\\+2\\6\end{pmatrix}=\begin{pmatrix}4\\+3\\7\end{pmatrix} \qquad \begin{pmatrix}30\\+6\\36\end{pmatrix}-\begin{pmatrix}20\\+1\\21\end{pmatrix}=\begin{pmatrix}10\\+5\\15\end{pmatrix}$$

$$\begin{array}{r} 8+5=13 \\ (-)\ 4+2=6 \\ \hline 4+3=7 \end{array} \qquad \begin{array}{r} 30+6=36 \\ (-)\ 20+1=21 \\ \hline 10+5=15 \end{array}$$

Or we might "add" addition tables:



Practice sheet 9 will reveal the pupil's grasp of the multiplication skills required by the usual long division algorithm.

EPS-9

Fill in the blanks to make true sentences. In each case, plan ahead so that the number listed over the squiggle (~~~) is as small as possible.

A.
1. 19 = 6 x 3 + 1
2. 47 = 11 x 4 + 3
3. 30 = 15 x 2 + 0
4. 60 = 8 x 7 + 4
5. 210 = 10 x 20 + 10
6. 585 = 11 x 50 + 35

B.
1. 36 = 3 x 10 + 6
2. 51 = 7 x 7 + 2
3. 100 = 3 x 30 + 10
4. 77 = 15 x 5 + 2
5. 400 = 16 x 25 + 0
6. 266 = 26 x 10 + 6

C.
1. 27 = 3 x 8 + 3
2. 88 = 9 x 9 + 7
3. 91 = 13 x 7 + 0
4. 131 = 13 x 10 + 1
5. 361 = 12 x 30 + 1
6. 457 = 45 x 10 + 7

D.
1. 150 = 10 x 12 + 2 x 12 + 6
2. 150 = 10 x 8 + 8 x 8 + 6
3. 150 = 20 x 7 + 1 x 7 + 3
4. 150 = 10 x 9 + 6 x 9 + 6

E.
1. 150 = 30 x 4 + 7 x 4 + 2
2. 150 = 10 x 11 + 3 x 11 + 7
3. 150 = 8 x 17 + 0 x 17 + 14
4. 150 = 6 x 23 + 0 x 23 + 12

F.
53 = 10 x 3 + 7 x 3 + 2

Your work area
53 = 30 + 21 + 2

G.
186 = 10 x 12 + 5 x 12 + 6
186 = 120 + 60 + 6

H.
366 = 20 x 17 + 1 x 17 + 9
366 = 340 + 17 + 9

I.
1251 = 20 x 42 + 9 x 42 + 33
1251 = 840 + 378 + 33

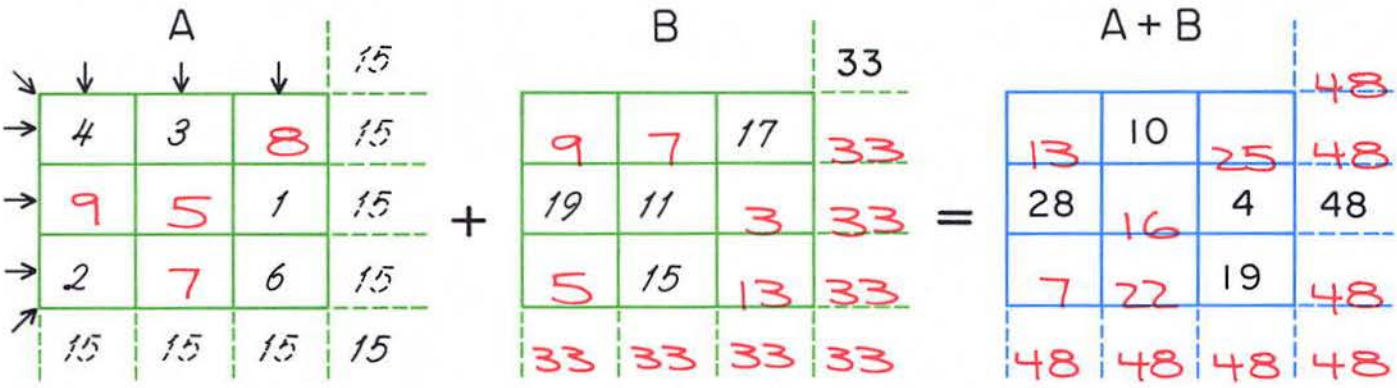
J.
After the Haskell family had driven at 60 miles per hour for three hours, they still had 45 miles left to go to the shore. The whole trip to the shore must have been 225 miles.

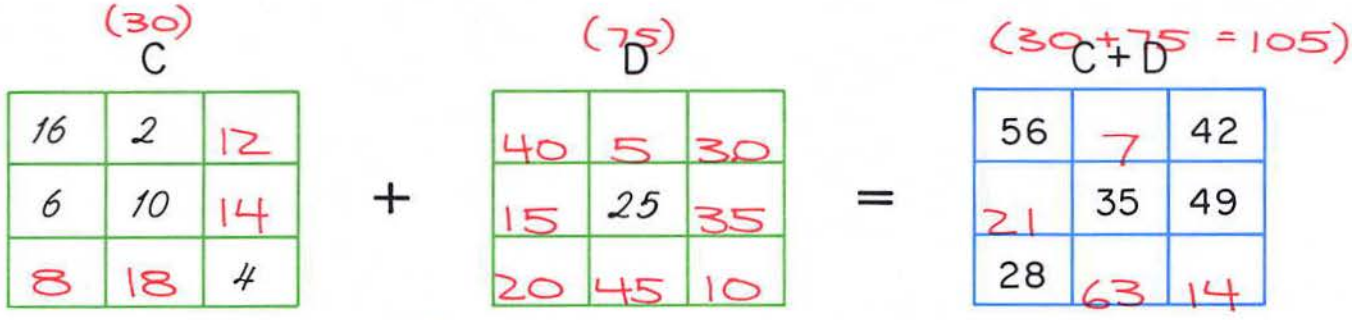
On the return trip, they took a longer route to visit a friend. The total trip was 500 miles. The route home was 275 miles. The route home was 50 miles more than the route they took on the way to the shore.

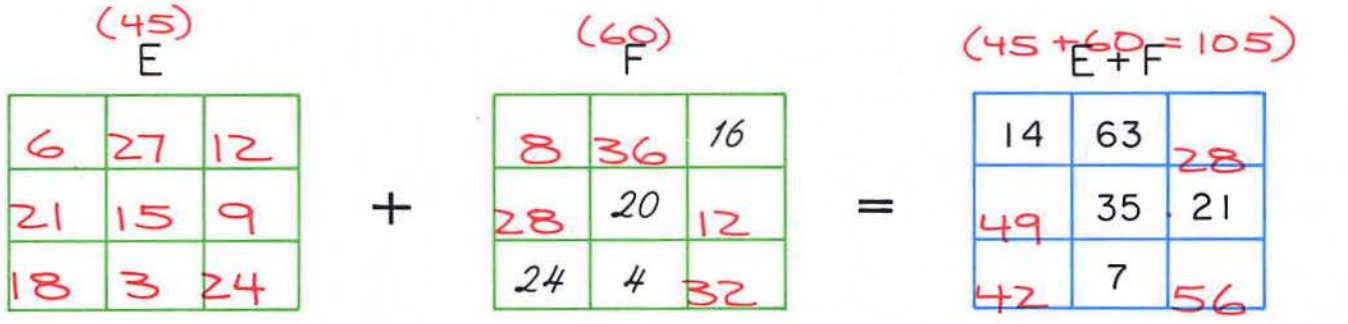
At 60 miles per hour, it would take 5 hours to drive 300 miles. At 50 miles per hour, it would take 6 hours to drive the same distance.

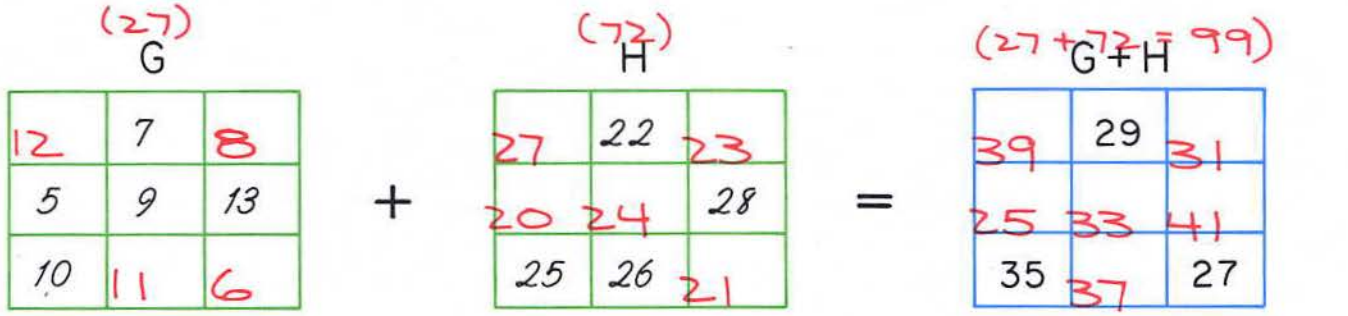
"I wish I could see inside Alec's machine," said Mr. Wilson. "Oh well, let's see what else it'll do."

Mr. Wilson moved the selector pointer to " — ." Magic Squares appeared where Cross Number Puzzle forms had been.









10 Using Alec's machine to add Magic Squares

Tasks: *Pupils study the responses of Alec Marson's machine to the input of* Magic Squares *and determine the rules of formation. They then complete all* Magic Squares.

Purpose: To provide experience in exploring structure and in discovering patterns. To provide meaningful and interesting practice in computation.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Even *Magic Squares* can be "added":

P				Q				P+Q		
7	10	1	+	12	13	5	=	19	23	6
0	6	12		3	10	17		3	16	29
11	2	5		15	7	8		26	9	13

Subtraction, as expected, requires planning. However, every example of adding (except doubling) suggests a pair of examples for subtraction. From the sum above (P+Q), we can subtract either of the addend *Magic Squares* (P or Q).

(P + Q)				Q				P		
19	23	6	−	12	13	5	=	7		
3	16	29		3	10	17				
26	9	13		15	7	8				

or

(P + Q)				P				Q		
19	23	6	−	7	10	1	=	12		
3	16	29		0	6	12				
26	9	13		11	2	5				

Thus, each example on pupil page 10 can lead to two examples of subtraction. The third example, E+F=(E+F), suggests this idea. No clues are provided about E. However, enough are provided to complete F and (E+F). Then the entries for E can be found by subtracting entries in F from corresponding entries in (E+F).

While the student's attention is directed always to exploration of a new idea, he is necessarily practicing his addition and subtraction skills. He is putting his arithmetic to work—using it to uncover new information, to test a theory, etc.

Some may be fascinated by the intricacies of incomplete *Magic Squares*. If so, here are some that can be completed in only one way.

11	13	3
1	9	17
15	5	7

32	1	21
7	18	29
15	35	4

3	0	3
2	2	2
1	4	1

3	1	2
1	2	3
2	3	1

7	14	0
0	7	14
14	0	7

11	25	6
9	14	19
22	3	17

These are difficult. Barely enough information is provided in each case. If they prove too difficult, provide an additional entry or two. A few may wish to invent their own *Magic Squares*—right from the beginning.

Practice sheet 10 will quickly sort out those whose computational skill is good enough to handle Level F without working through Level E.

Glance through the page before you begin. Complete the easier exercises first. If you have the time, try to finish the more difficult exercises as well.

EPS-10

1. 3714 + 4986 = 8700

2. 6054 − 1095 = 4959

3. $3\overline{)3745} = 1248\frac{1}{3}$

4. 12,807 × 8 = 102456

5. $14.54 + 7.80 + 43.67 + 18.83 + .68 ≠ $85.52

6. $35\overline{)4830} = 138$ (35, 133, 105, 280, 280, 0)

7. 357 × 824 = 1428 + 714 + 2856 = 294168

8. 10,046 − 678 = 9,368

9. 158.45 × 7 = 1109.15

10. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

11. $\frac{2}{3} \times \frac{3}{5} = \frac{2}{5}$

12. $1\frac{3}{4} \times 3 = 5\frac{1}{4}$

13. $\frac{7}{8} \div 2 = \frac{7}{16}$

14. $1\frac{2}{3} \times 2\frac{3}{4} = 4\frac{7}{12}$

15. $\frac{5}{6} \div \frac{2}{3} = 1\frac{1}{4}$

16. $1\frac{5}{6} + 3\frac{7}{8} = 5\frac{17}{24}$

17. $4\frac{2}{3} \div 2\frac{1}{5} = 2\frac{4}{33}$

18. .5 + .3 = .8

19. .8 + 3 = 3.8

20. 1.25 + 2.5 = 3.75

21. 3.1 + .13 = 3.23

22. 1.7 − .8 = .9

23. 3 − 1.3 = 1.7

24. 2.1 − .007 = 2.093

25. .075 − .05 = .025

26. .3 × .4 = .12

27. 1.5 × 8 = 12

28. 2.17 × .03 = .0651

29. .08 × .09 = .0072

30. 2.7 ÷ 3 = .9

31. 42 ÷ .6 = 70

32. 1.44 ÷ 1.2 = 1.2

33. .7 ÷ .007 = 100

34. 10% × 80 = 8

35. 25% of 7 = $1\frac{3}{4}$

36. 17% × 10 = 1.7

37. 135% × 3 = 4.05

38. $15\overline{).675} = .045$

39. $.42\overline{)562.8} = 1340.$

40. 3.24 × 29 = 9.396

41. $3^2 + 2^3 = 17$

42. $\sqrt{25} + \sqrt{64} = 13$

11 Using Alec's machine to multiply Cross Number Puzzles by whole numbers

Mr. Wilson moved the selector to " + ." "Ah-hah!" he said aloud, "now I can work on Cross Number Puzzle multiplication."

A.

100	30	4	
		5	385
100	40		
500		15	

x2 =

600	160		
		12	
1,000			

B.

100		3	
		6	

x5 =

500	450		995
	100		615
500			780
1,500	800		

C.

	40	5	
		8	
	120	15	735

x3 =

600			
	60		
900			

D.

200	20		
300	60		
200	60		
			856

x6 =

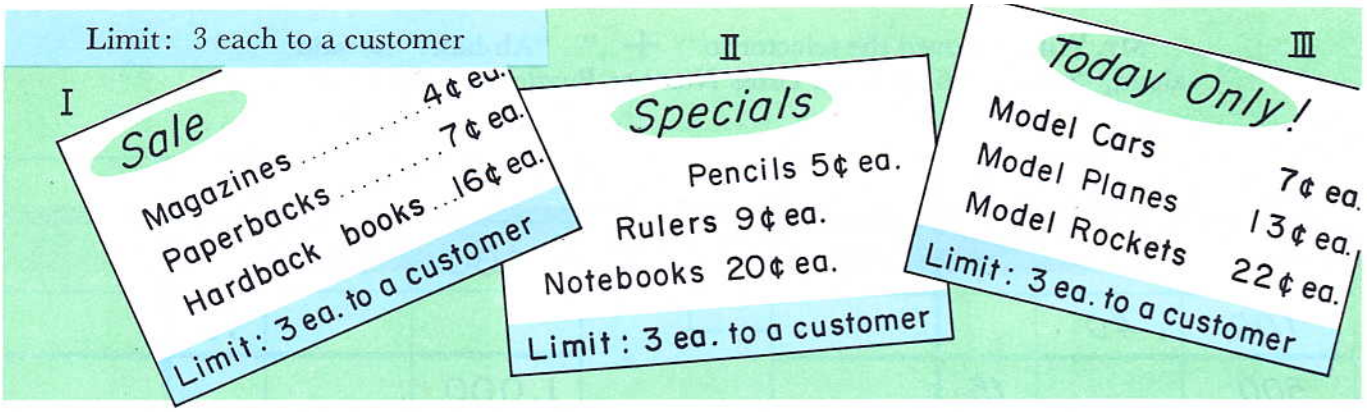
		54	
1,200	360	30	
4,200	840		

E.

1,000	70	8	
1,500	110	21	

x4 =

400		16	
	160		



These signs refer to 3 different sales. We shall refer to them as Sale I, Sale II, and Sale III.

Could you spend exactly 36¢ under the terms of Sale II? The limit would rule out a purchase of 4 rulers at 9¢ each. A notebook costs 20¢. Since no combination of pencils and rulers costs 16¢, 36¢ could not be spent. The largest purchase without including a notebook is 3 pencils and 3 rulers for 42¢. One less pencil would make a purchase of _____¢. Three pencils and 2 rulers would cost _____¢.

It seems that exactly 36¢ can not be spent under the terms of Sale II.

Could exactly 36¢ be spent at Sale I? Yes: _____ magazines, _____ paperbacks, and _____ hardback books.

Could exactly 36¢ be spent at Sale III? Yes: _____ cars, _____ planes, _____ rockets.

Spend exactly 39¢ at each sale.

Sale I:

_____ magazines, _____ paperbacks, _____ hardbacks

Sale II:

_____ pencils, _____ rulers, _____ notebooks

Sale III:

_____ cars, _____ planes, _____ rockets

Can exactly 39¢ be spent in any other way at any one of the sales? _____

Spend exactly 53¢ at each sale.

Sale I:

_____ magazines, _____ paperbacks, _____ hardbacks

Sale II:

_____ pencils, _____ rulers, _____ notebooks

Sale III:

_____ cars, _____ planes, _____ rockets

Spend exactly 62¢ at each sale.

Sale I:

_____ magazines, _____ paperbacks, _____ hardbacks

Sale II:

_____ pencils, _____ rulers, _____ notebooks

Sale III:

_____ cars, _____ planes, _____ rockets

In any one of the sales, can exactly 53¢ or 62¢ be spent in more than one way? _____

In any one of the sales, can an exact amount be spent in more than one way? _____

If you answered yes, list several examples.

_____, _____, _____

_____, _____, _____

The largest purchase at each sale.

Sale I: _____ Sale II: _____ Sale III: _____

The number of different purchases at a sale.

At Sale I, how many different purchases of 1 or more items could you make? _____. Is the same true of Sales II and III? _____

A.

10	−	3	= 7
−	+	−	
4	+	2	= 6
=	=	=	
6	−	5	= 1

B.

15	−	8	= 7
+	+	+	
5	−	2	= 3
=	=	=	
20	−	10	= 10

C.

2	x	6	= 12
x	÷	x	
9	÷	3	= 3
=	=	=	
18	x	2	= 36

D.

8	x	6	= 48
÷	÷	÷	
4	x	2	= 8
=	=	=	
2	x	3	= 6

Watch the Signs!

Study the examples above. WHAT ARE MY RULES?

1.

9	+	16	=
+	−	+	
12	−	7	=
=	=	=	
	+		=

2.

24	+	16	=
−	−	−	
8	+	7	=
=	=	=	
	+		=

3.

	+		= 55
−	+	−	
27	−	19	=
=	=	=	
	+	31	=

4.

5	x	6	=
x	÷	x	
12	÷	3	=
=	=	=	
	x		=

5.

36	x	2	=
÷	x	÷	
9	÷	3	=
=	=	=	
	x		=

6.

	x	4	=
÷	÷	÷	
	x		= 12
=	=	=	
4	x		= 8

7.

52	−	18	=
−	−	−	
	−		= 16
=	=	=	
27	−		=

8.

30	+		= 38
+	+	+	
	+	7	=
=	=	=	
	+		= 95

9.

	x		=
x	x	x	
	x	2	= 6
=	=	=	
12	x		= 120

Please put in the signs and complete each example.

10.

31		18	=
24		15	=
=	=	=	
			= 4

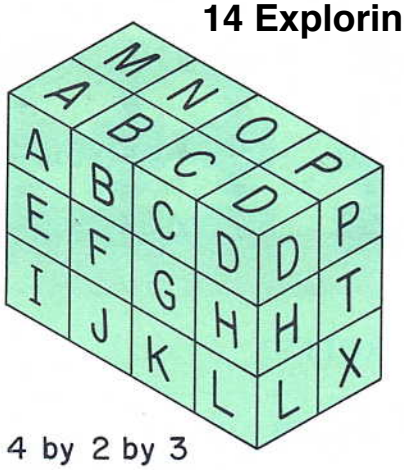
11.

40			=
		7	=
=	=	=	
		18	= 42

12.

75		30	=
			= 5
=	=	=	
91			=

14 Exploring products of three factors



4 by 2 by 3

This is a 4 by 2 by 3 stack of blocks. You see exactly 3 faces of the D block, exactly 2 faces of the A, B, C, H, L, and P blocks. You see exactly one face of the E, F, G, I, J, K, M, N, O, T, and X blocks. There are 6 blocks you cannot see at all. If we follow the lettering scheme, these blocks would be the Q, R, S, U, V, and W blocks.

Here is a summary of this information:

	Exact number of faces showing				
	3	2	1	0	Total
4 by 2 by 3	1	6	11	6	24

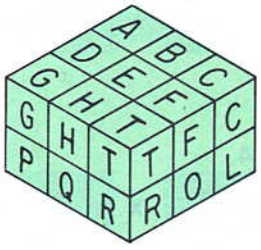
Number of blocks

Make similar records for the stacks shown below and at the right.

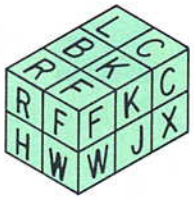
	Exact number of faces showing				
	3	2	1	0	Total
4 by 2 by 3					
2 by 2 by 2					
3 by 3 by 2					
2 by 3 by 2					
3 by 5 by 2					
3 by 3 by 3					
3 by 4 by 8					
5 by 5 by 5					
4 by 4 by 4					
7 by 2 by 2					



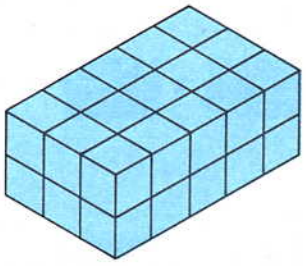
2 by 2 by 2



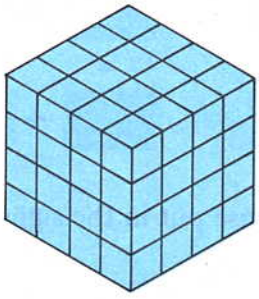
3 by 3 by 2



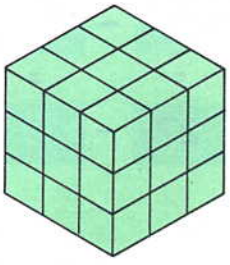
2 by 3 by 2



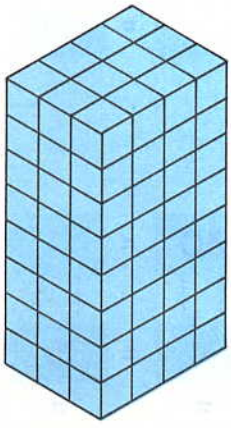
3 by 5 by 2



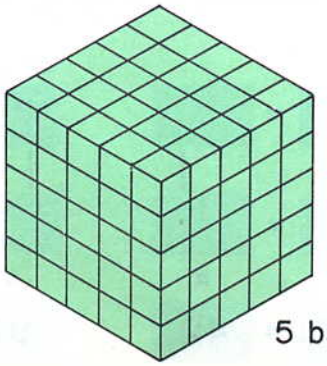
4 by 4 by 4



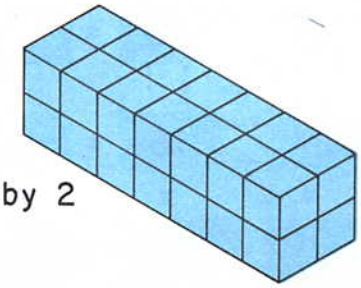
3 by 3 by 3



3 by 4 by 8



5 by 5 by 5



7 by 2 by 2

X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0		
1	0	1	2	3	4	5	6	7	8	9			
2	0	2	4	6	8								
3	0	3	6	9									
4	0	4	8	12							0		
5	0	5	10				30					55	
6	0	6	12										
7	0	7											
8	0	8											
9	0	9											
10	0	10											
11	0	11											
12	0	12											
13	0	13											
14	0	14											
15	0	15											
16	0	16											
17	0	17											
18	0	18											
19	0	19											
20	0	20			80								
21	0	21											
22	0	22											
23	0	23											
24	0												
25													
26													
27													
28													
29													
30				90									

15 Using part of the table to determine some missing products

Mr. Dawson's Class Tackles a BIG JOB

"At first it was fun to learn how to multiply and divide," Marge began, "but now it's getting a little boring. I wish we had a computer."

Everyone in the class agreed.

"Well, why not make one," Mr. Dawson suggested.

"How?" everyone wanted to know.

"We can build a big multiplication table that will contain the product of any pair of 1- or 2-digit numbers from $0 \times 0 = 0$ to $99 \times 99 =$ _____."

"Great! When can we get started?"

With Mr. Dawson's help, the class tackled this big job.

Part of the table is shown on this page. Study the plan of the table. Please fill in the missing numbers.

Please use the table to complete the following sentences:

4 x 5 = ____	3 x 29 = ____
12 x 3 = ____	4 x ____ = 56
17 x 5 = ____	____ x 5 = 90
4 x 16 = ____	15 x ____ = 60
5 x 23 = ____	29 x ____ = 145
27 x 4 = ____	____ x 19 = 76

LOOKING FOR SHORTCUTS

"How many entries will there be in our table?" Mr. Dawson asked.

"Well," Ben thought out loud, "there will be 100 entries in each row and there will be 100 rows."

100 x 100 = _____

"That's too many multiplications," Marge protested.

Mr. Dawson said, "Let's work on one corner of the table and talk about our results."

X	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	(3)	4	5
2	0	2	4	(6)	8	(10)
3	0	(3)	(6)	9	12	15
4	0	4	8	12	16	(20)
5	0	5	(10)	15	(20)	25
	↑ a	↑ b	↑ c			↑ d

"The diagonal of the table contains the familiar group of numbers called 'square numbers.' We'll talk about them later. But notice the entries on one side of the diagonal are just like those on the other. This is not surprising since we know that

3 x 1 = 1 x 3 2 x 3 = 3 x 2
2 x 5 = 5 x 2 4 x 5 = 5 x 4
etc.

"So, we can leave one side of the diagonal blank."

"Sure," Marge agreed. "On the last page, I looked up all the answers I needed in the part shaded in blue."

Jane had been working. She spoke up: "There are 10,000 less 100 that are not on the diagonal. Half of that number is _____."

Mr. Dawson had more words of encouragement. "Notice the column I've labeled 'a' — it's all zeros; and the column I've labeled 'b' is just a list of the whole numbers.

"I think I know why you point out the columns labeled 'c' and 'd'," Helen volunteered. "We learned how to count by 2's and 5's a long time ago. Including the one on the diagonal and those below it, there are _____ multiples of 2 and _____ multiples of 5 we can compute as fast as we can write."

Interest in working on the table increased as everyone began to see ways to make the job easier. Mr. Dawson had some words of advice before they started.

"Our first job will be to find shortcuts to avoid unnecessary computations.

"Our second job will be to look for quick ways to check our results. We don't want careless errors to get into our table.

"Finally, we will plan a way to share the work to avoid too much repetition."

Mark's Chain Reaction Idea

"I was experimenting with making a multiplication table.

"I notice that I could start with a couple of entries — and they would lead me to another. Here are two rows."

A.

X	2	3 → 5 → 8 → 13 →	→					(beyond chart)
7	14	21 → 35 → 56						
9	18	27 → 45						

Top of columns: 2 + 3 = 5 ; 3 + 5 = 8 ; 5 + 8 = 13 ; 8 + 13 = ____ ; 13 + ____ = ____

Row 7: 14 + 21 = 35 ; 21 + 35 = 56 ; 35 + 56 = ____ ; 56 + ____ = ____

Row 9: 18 + 27 = 45 ; 27 + 45 = ____ ; 45 + ____ = ____ ; ____ + ____ = _____

"Then I noticed that I could add the entries in these two rows and get entries for row 16 because 7 + 9 = 16."

B.

	2	3	5	8	13	21				(beyond chart)
16	32	48								

14 + 18 = ____ ; 21 + 27 = ____ ; 35 + 45 = ____ ; 56 + ____ = _____

"Here are some more examples of my two-way Chain Reaction."

C.

X	7	10	17	27	44
12					
13					
25					
38					
63					

D.

X	20	3	23	26	49
9					
10					
19					
29					
48					

E.

X	7	20	27
5			
30			
35			

F.

X	11	22	33
6			
12			
18			

G.

X	8	16	24
9			
18			
27			

H.

X	12	10	22
12			
24			
36			

I.

X	22	10	32
36			
10			
46			

J.

X	32	2	34
46			
4			
50			

18 Paul's "Double or Add" system

Paul's "Double or Add" System

"I use another kind of Chain Reaction that gives me every entry in a column," said Paul. "I double an entry or add a pair of neighbors. I'll use the column under 39 to show you my system.

"To get the entry in row 2, I DOUBLE the entry in row 1. To get the entry for row 3, I ADD THE NEIGHBORING entries in rows 1 and 2, etc."

I.

	39	
1	39	
2	78	.... double the entry in row 1
3	117	.... add the entries in rows 1 and 2
4	156	.... double the entry in row 2
5	195	.... add the entries in rows 2 and 3
6		.... double the entry in row 3
7		.... add the entries in rows 3 and 4
8		.... double the entry in row 4
9		.... add the entries in rows 4 and 5
10		.... double the entry in row 5

"At this point, I use what I already know to enter some checkpoints."

II.

1 x 39 = 39 then 10 x 39 = 390

2 x 39 = _____ then 20 x 39 = _____

3 x 39 = _____ then 30 x 39 = _____

4 x 39 = _____ then 40 x 39 = _____

5 x 39 = _____ then 50 x 39 = _____

etc.

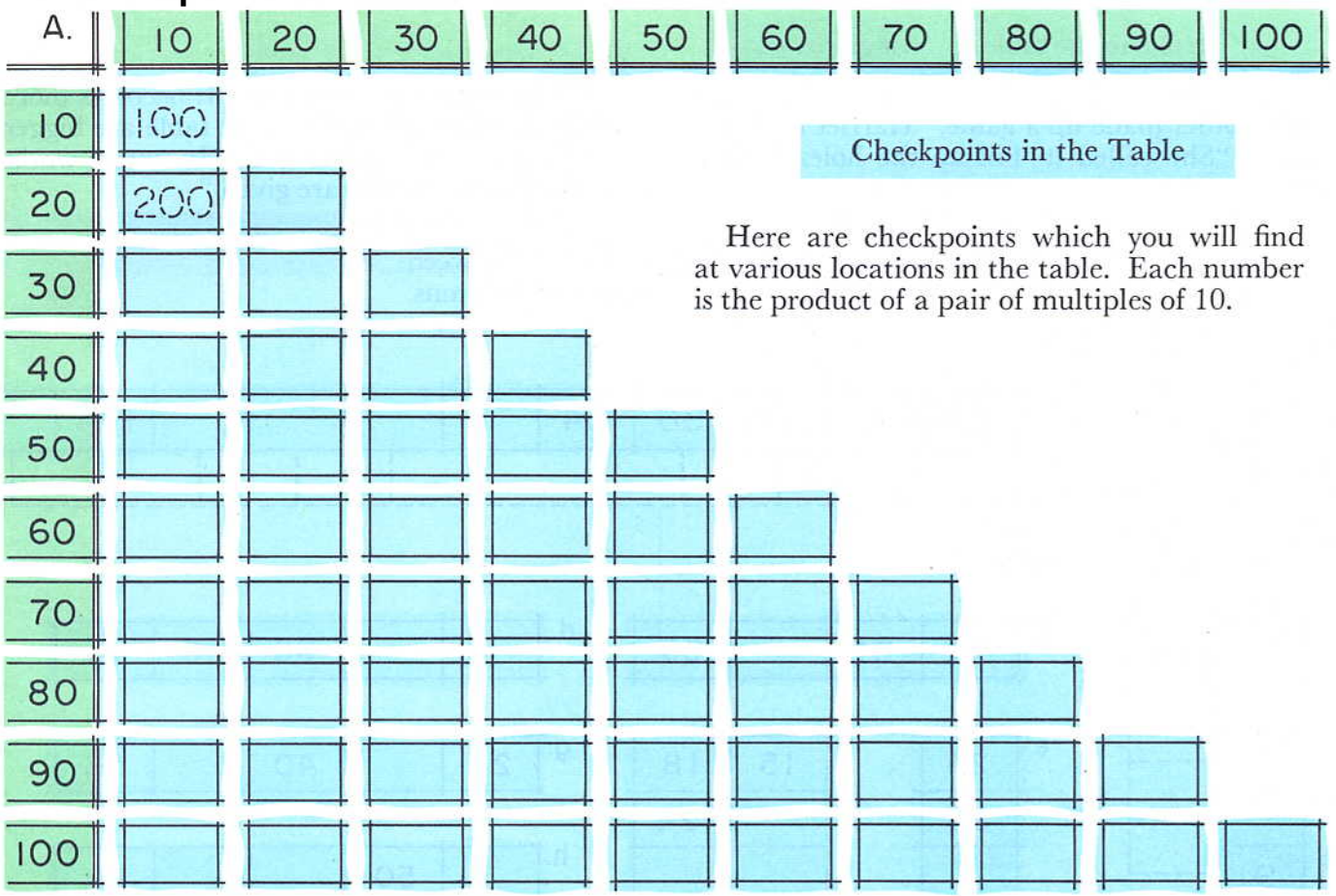
Please enter the checkpoints and try using Paul's system.

III.

	39	
11	429	R5 and R6
12	468	double R6
13	507	R6 and R7
14		
15		
16		
17		
18		
19		Check-point ✓
20	780 ✓	
21		R10 and R11
22		
23		
24		
25		
26		
27		
28		
29		
30	1,170 ✓	
31		
32		
33		
34		
35		
36		
37		
38		
39		
40	✓	

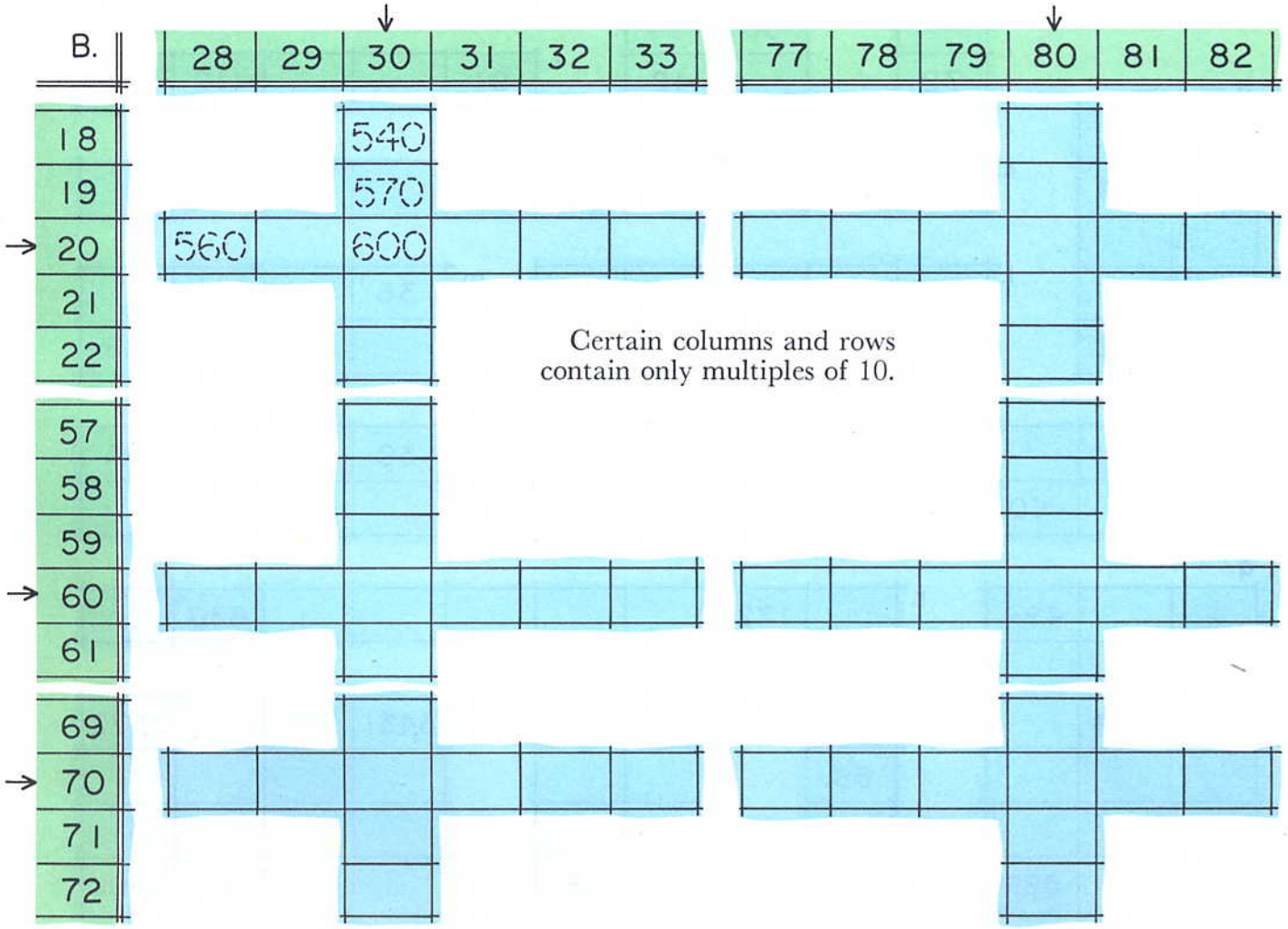
IV.

	39	
41		
42		
43		
44		
45		
46		
47		
48		
49		
50		✓
82		
83		
84		
85		
86		
87		
88		
89		
90		✓
91		
92		
93		
94		
95		
96		
97		
98		
99		
100		✓



Checkpoints in the Table

Here are checkpoints which you will find at various locations in the table. Each number is the product of a pair of multiples of 10.



Certain columns and rows contain only multiples of 10.

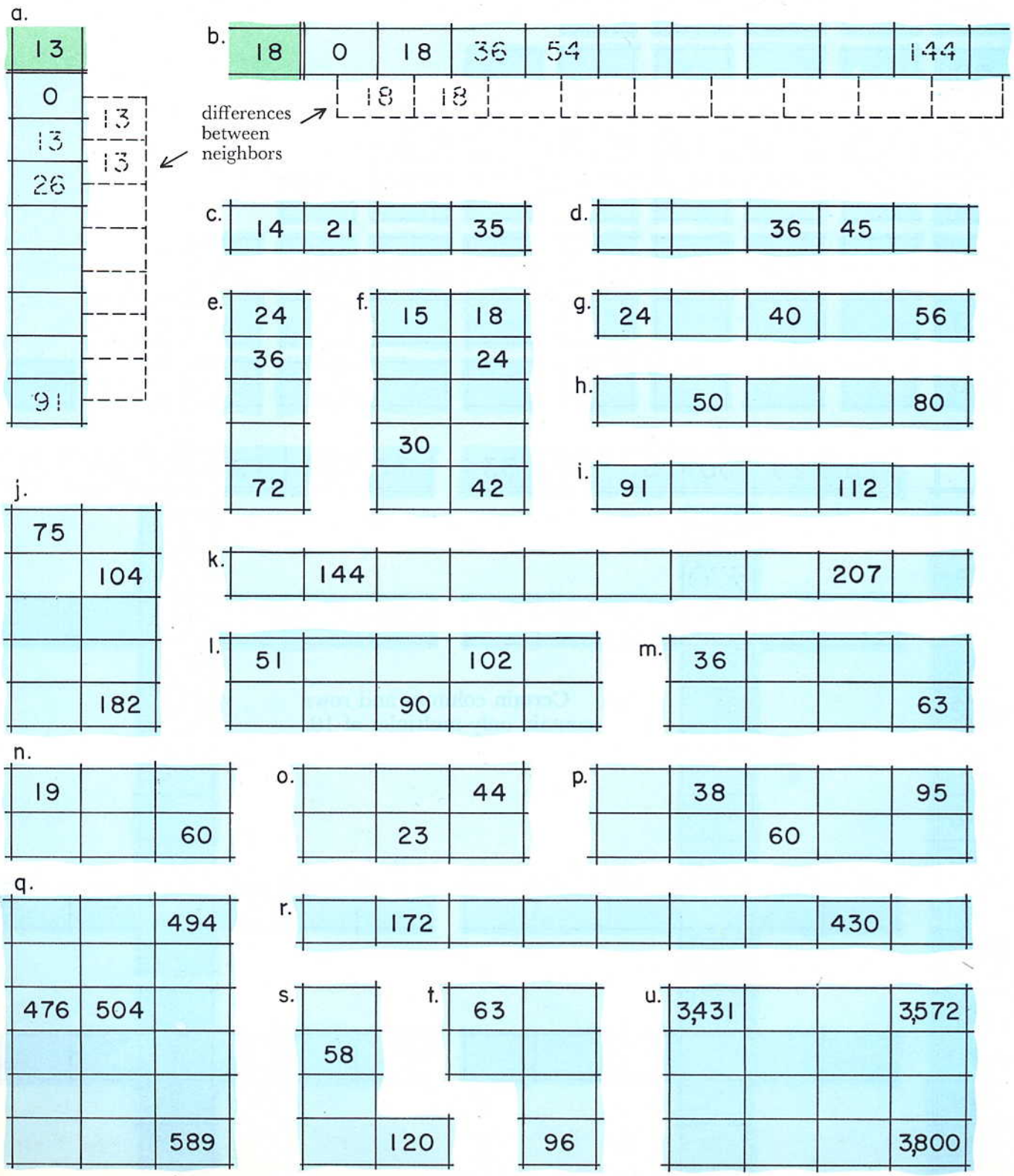
20 Holes in the table ... a game

"Holes in the table". . . a game.

"Mother made up a game," Harriet told the class. "She called it 'Fill up the holes in the table.'

"The game starts out easy. It becomes more difficult when the holes in the table are bigger and the numbers are larger. My system works so long as two entries are given."

Hint: Notice the differences between neighbors in the rows and columns.



21 Ruth studies the diagonal (square numbers)

Ruth Studies the Diagonal

A.

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0												
1		1											
2			4										
3				9									
4													
5													
6													
7													
8													
9													
10													
11													
12													

Ruth found these "differences between neighbors"

etc.

"I found a chain reaction along the diagonal," said Ruth.

"I noted the differences between neighbors."

1 − 0 = ____ 9 − 4 = ____

4 − 1 = ____ 16 − 9 = ____

25 − 16 = ____

36 − 25 = ____

etc.

"My sister said that 1^2, 2^2, 3^2, 4^2, etc., is shorthand for 1×1, 2×2, 3×3, 4×4, etc. She said the shorthand is read 'one squared, two squared, three squared, four squared.'

"My tables look like this:"

B.

0 x 0	=	0^2	=	0
1 x 1	=	1^2	=	1
2 x 2	=	2^2	=	4
3 x 3	=	3^2	=	9
4 x 4	=	4^2	=	
5 x 5	=	5^2	=	
6 x 6	=	6^2	=	
7 x 7	=	7^2	=	
8 x 8	=	8^2	=	
9 x 9	=	9^2	=	

1, 3, 5

C.

10^2 =	✓
11^2 =	
12^2 =	
13^2 =	
14^2 =	
15^2 =	
16^2 =	
17^2 =	
18^2 =	
19^2 =	

D.

20^2 =	✓
21^2 =	
22^2 =	
23^2 =	
24^2 =	
25^2 =	
26^2 =	
27^2 =	
28^2 =	
29^2 =	

"I noticed a pattern for the units digits in each row of my table. I use this pattern to check my work."

$6^2 = 36$ $16^2 = 256$ $26^2 = 676$

$3^2 = 9$ $13^2 = 169$ $23^2 = 529$ etc.

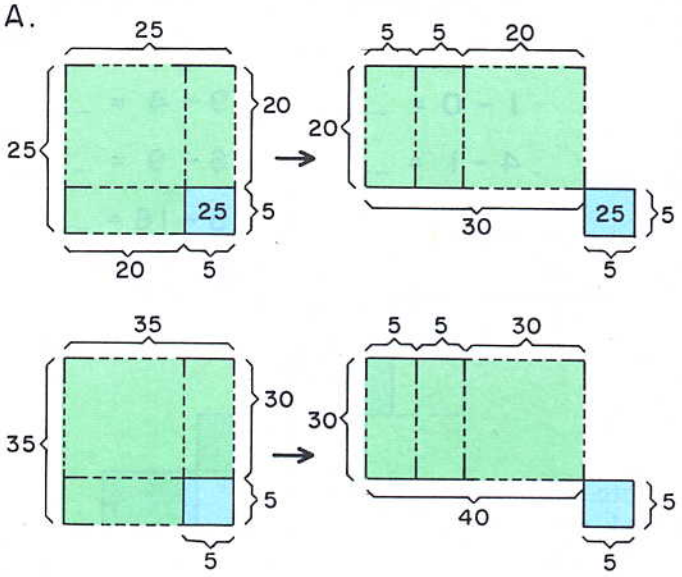
22 Mr. Dawson's hints: (a+b)^2 = a^2 + 2ab + b^2 and [(a+b) +1]^2 - (a+b)^2 = 2(a+b) + 1

Mr. Dawson's Hints

"You can easily compute the squares of multiples of 10 — 10^2, 20^2, 30^2, 40^2, etc.

"Here are some sketches that may help you find a shortcut for computing squares of the following multiples of 5 — 15^2, 25^2, 35^2, 45^2, etc."

A.



B.

15 × 15 = 225	25 × 25	35 × 35
45 × 45	55 × 55	65 × 65

"Can you find a rule to help you check the differences between neighbors?"

C.

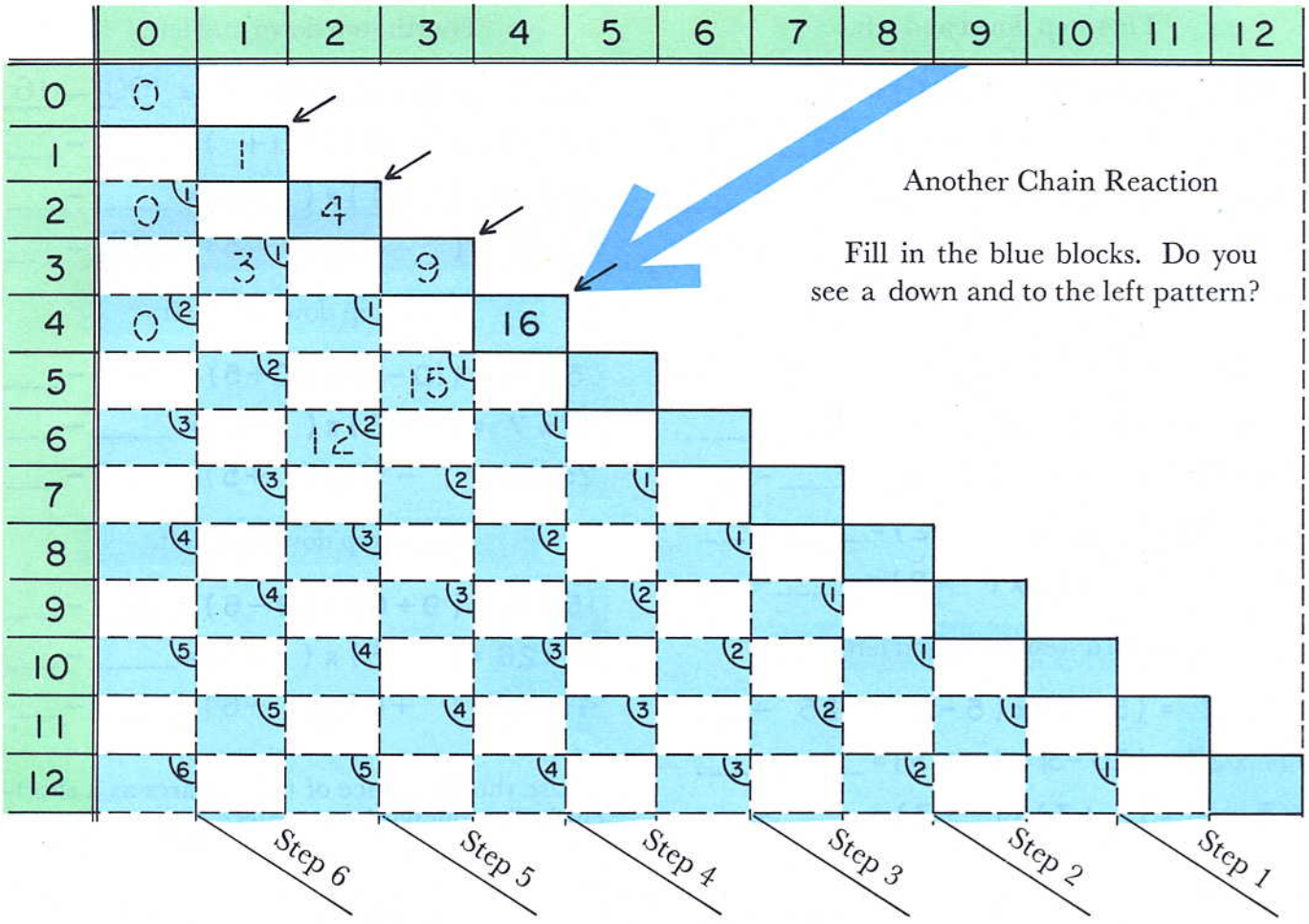
6^2 = 36	13	25^2 =	
7^2 = 49		26^2 =	
12^2 =		40^2 =	
13^2 =		41^2 =	
20^2 =		17^2 =	
21^2 =		18^2 =	

D.

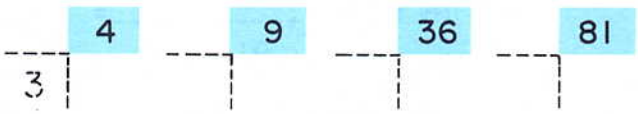
30^2 = 0 ✓	61	
31^2 = 1		
32^2 = 4		
33^2 = 9		
34^2 = 6		
35^2 = 5 ✓		
36^2 = 6		
37^2 = 9		
38^2 = 4		
39^2 = 1		
40^2 = 0 ✓		
41^2 =		
42^2 =		
43^2 =		
44^2 =		
45^2 = ✓		
46^2 =		
47^2 =		
48^2 =		
49^2 =		
50^2 = ✓		
51^2 =		
52^2 =		
53^2 =		
54^2 =		
55^2 = ✓		
56^2 =		
57^2 =		
58^2 =		
59^2 =		
60^2 = ✓		
61^2 =		
62^2 =		
63^2 =		
64^2 =		

E.

65^2 = ✓	
66^2 =	
67^2 =	
68^2 =	
69^2 =	
70^2 = ✓	
71^2 =	
72^2 =	
73^2 =	
74^2 =	
75^2 = ✓	
76^2 =	
77^2 =	
78^2 =	
79^2 =	
80^2 = ✓	
81^2 =	
82^2 =	
83^2 =	
84^2 =	
85^2 = ✓	
86^2 =	
87^2 =	
88^2 =	
89^2 =	
90^2 = ✓	
91^2 =	
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94^2 =	
95^2 = ✓	
96^2 =	
97^2 =	
98^2 =	
99^2 =	

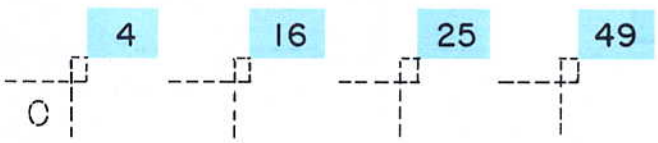


The first step down and to the left:



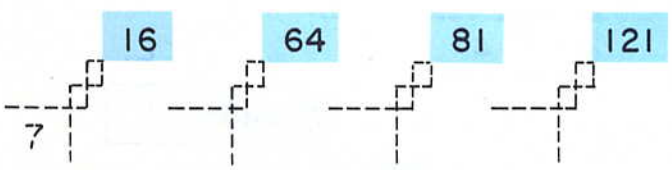
The entry at step 1 is _____ less than the entry on the diagonal.

The second step down and to the left:



The entry at step 2 is _____ less than the entry on the diagonal.

The third step down and to the left:



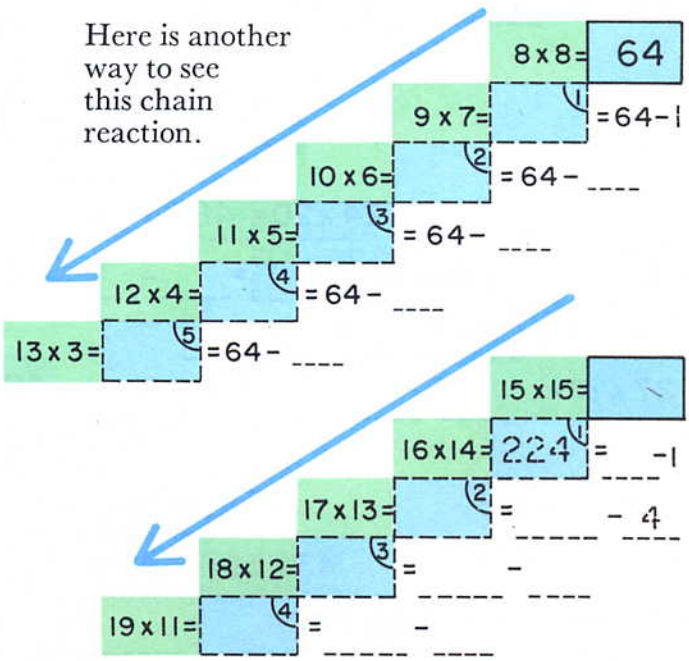
The entry at step 3 is _____ less than the entry on the diagonal.

The entry at step 4 is _____ less than the entry on the diagonal.

The entry at step 5 is _____ less than the entry on the diagonal.

The entry at step 6 is _____ less than the entry on the diagonal.

Here is another way to see this chain reaction.



First step down and left.

$7 \times 5 = (6+1) \times (6-1) = 36 - \_\_\_$

$10 \times 8 = (\ +1) \times (\ -1) = \_\_\_\_\_ - \_\_\_$

$21 \times 19 = (\ +1) \times (\ -1) = \_\_\_\_\_ - \_\_\_$

$24 \times 26 = (\ -1) \times (\ +1) = \_\_\_\_\_ - \_\_\_$

$49 \times 51 = (\ -1) \times (\ +1) = \_\_\_\_\_ - \_\_\_$

Second step down and left.

$10 \times 6 = (8+2) \times (8-2) = 64 - \_\_\_$

$13 \times 17 = (15-2) \times (15+2) = \_\_\_\_\_ - \_\_\_$

$32 \times 28 = (\ +2) \times (\ -2) = \_\_\_\_\_ - \_\_\_$

$47 \times 43 = (\ +2) \times (\ -2) = \_\_\_\_\_ - \_\_\_$

Third step down and left.

$8 \times 2 = (5+3) \times (5-3) = 25 - \_\_\_$

$17 \times 23 = (20-3) \times (\ +3) = \_\_\_\_\_ - \_\_\_$

$15 \times 9 = (\ +3) \times (\ -3) = \_\_\_\_\_ - \_\_\_$

Fourth step down and left.

$14 \times 6 = (10+4) \times (10-4) = 100 - 16$

$7 \times 15 = (11-4) \times (11+\ ) = \_\_\_\_\_ - \_\_\_$

$21 \times 29 = (\ -4) \times (\ +4) = \_\_\_\_\_ - \_\_\_$

$11 \times 3 = (\ +4) \times (\ -4) = 49 - \_\_\_$

Fifth step down and left.

$5 \times 15 = (10-5) \times (10+5) = 100 - \_\_\_$

$17 \times 7 = (\ +5) \times (\ -5) = \_\_\_\_\_ - \_\_\_$

$25 \times 15 = (\ -5) \times (\ +5) = \_\_\_\_\_ - \_\_\_$

Sixth step down and left.

$15 \times 3 = (9+6) \times (9-6) = 81 - \_\_\_$

$14 \times 26 = (\ -6) \times (\ +6) = \_\_\_\_\_ - \_\_\_$

$41 \times 29 = (\ +6) \times (\ -6) = \_\_\_\_\_ - \_\_\_$

Use the difference of two squares as a short-cut in the following examples:

A.

$25 \times 25 = \square$

$26 \times 24 = \square = \_\_\_\_ - 1$

$27 \times 23 = \square = \_\_\_\_ - 4$

$28 \times 22 = \square = \_\_\_\_ - \_\_\_$

$29 \times 21 = \square = \_\_\_\_ - \_\_\_$

$30 \times 20 = 600 = 625 - 25$

B.

$11 \times 9 = \_\_\_\_\_ - \_\_\_\_ = \_\_\_\_\_$

$29 \times 31 = \_\_\_\_\_ - \_\_\_\_ = \_\_\_\_\_$

$16 \times 14 = \_\_\_\_\_ - \_\_\_\_ = \_\_\_\_\_$

$11 \times 7 = 81 - 4 = \_\_\_\_\_$

$18 \times 22 = \_\_\_\_\_ - \_\_\_\_ = \_\_\_\_\_$

$12 \times 18 = \_\_\_\_\_ - \_\_\_\_ = \_\_\_\_\_$

$16 \times 8 = \_\_\_\_\_ - \_\_\_\_ = \_\_\_\_\_$

$35 \times 45 = \_\_\_\_\_ - \_\_\_\_ = \_\_\_\_\_$

$18 \times 6 = \_\_\_\_\_ - \_\_\_\_ = \_\_\_\_\_$

C.

$22 \times 22 = \square$

$23 \times 21 = \square = \_\_\_\_ - \_\_\_$

$24 \times 20 = 480 = \_\_\_\_ - 4$

$25 \times 19 = \square = \_\_\_\_ - \_\_\_$

$26 \times 18 = \square = \_\_\_\_ - \_\_\_$

$27 \times 17 = \square = \_\_\_\_ - \_\_\_$

D.

$39 \times 39 = \square$

$40 \times 38 = \square = \_\_\_\_ - \_\_\_$

$41 \times 37 = \square = \_\_\_\_ - \_\_\_$

$42 \times 36 = \square = \_\_\_\_ - \_\_\_$

$43 \times 35 = \square = \_\_\_\_ - \_\_\_$

$44 \times 34 = \square = \_\_\_\_ - \_\_\_$

E.

$45 \times 45 = \square$

$46 \times 44 = \square = \_\_\_\_ - \_\_\_$

$47 \times 43 = \square = \_\_\_\_ - \_\_\_$

Completing CHUNKS of the TABLE . . . or, finding shortcuts

A.

X	5	6	7	9	10	11	15	16	17
6	30	36							
7	35	42							
8	40	48	56						
9									
10		60	70						
11	55								
16									
17									
18									

$$\begin{array}{r} 6 \\ \times\ 7 \end{array} \qquad \begin{array}{r} 10 \\ \times\ 7 \end{array}$$

$$\begin{array}{r} 16 \\ \times\ 7 \end{array}$$

$$\begin{array}{r} 6 \\ \times\ 10 \end{array} \qquad \begin{array}{r} 10 \\ \times\ 10 \end{array}$$

$$\begin{array}{r} 16 \\ \times\ 10 \end{array}$$

$$\begin{array}{r} 16 \\ \times\ 17 \\ \hline 112 \\ 160 \\ \hline 272 \end{array}$$

B.

X	7	8	9	19	20	21	27	28	29
4									
5									
6									
29									
30									
31									
34									
35									
36									

$$\begin{array}{r} 8 \\ \times\ 5 \end{array} \qquad \begin{array}{r} 20 \\ \times\ 5 \end{array}$$

$$\begin{array}{r} 28 \\ \times\ 5 \end{array}$$

$$\begin{array}{r} 8 \\ \times 30 \end{array} \qquad \begin{array}{r} 20 \\ \times 30 \end{array}$$

$$\begin{array}{r} 28 \\ \times 30 \end{array}$$

$$\begin{array}{r} 28 \\ \times\ 35 \\ \hline \\ 0 \\ \hline \end{array}$$

Use the table to save time in finding the missing numbers below:

$$\begin{array}{r} 16 \\ \times\ 7 \end{array} \qquad \begin{array}{r} 29 \\ \times 30 \end{array} \qquad \begin{array}{r} 36 \\ \times\ \quad \\ \hline 324 \end{array} \qquad \begin{array}{r} 34 \\ \times 27 \end{array} \qquad \begin{array}{r} 27 \\ \times\ \quad \\ \hline 918 \end{array} \qquad \begin{array}{r} 31 \\ \times\ \quad \\ \hline 899 \end{array} \qquad \begin{array}{r} \times\ \quad \\ \hline 256 \end{array} \qquad \begin{array}{r} \times\ \quad \\ \hline 841 \end{array}$$

III Building and Exploring the 0 x 0 to 99 x 99 Multiplication Table

X	7	8	9	29	30	31	32
5	35	40	45	145	150	155	160
6	42	48	54	174	180	186	192
7	49	56	63	203	210	217	224
48	336	384	432	1,392	1,440	1,488	1,536
49	343	392	441	1,421	1,470	1,519	1,568
50	350	400	450	1,450	1,500	1,550	1,600
51	357	408	459	1,479	1,530	1,581	1,632

Fill in the missing digits. Use the completed table above if it will save you some time.

a. 7 x 30 = 210

b. 21 x 40 = 840

c. 25 x 30 = 750

d. 9 x 70 = 630

e. 32 x 7 = 224

f. 90 x 7 = 630

g. 9 x 7 = 63

h. 30 x 5 = 150

i. 12 x 6 = 72

j. 7 x 8 = 56

k. 29 x 5 = 145

l. 9 x 8 = 72

m. 7 x 7 = 49

n. 31 x 50 = 1550

o. 32 x 6 = 192

p. 33 x 6 = 198

q. 16 x 54 = 864

r. 23 x 28 = 644

s. 16 x 53 = 848

t. 55 x 55 = 3025

u. 9 x 6 = 54

v. 8 x 4 = 32

w. 8 x 8 = 64

x. 4 x 9 = 36

y. 17 x 18 = 306

z. 29 x 49 = 1421

A. 64 x 19 = 1216

B. 32 x 51 = 1632

C. 29 x 49 = 1421

D. 48 x 32 = 1536

E. 64 x 29 = 1856

F. 85 x 47 = 3995

G. 34 x 78 = 2652

26 Fill in the missing digits ... another game

Tasks: *Pupils complete the multiplication grid and reconstruct examples of multiplication in the conventional form, using the completed grid to save time.*

Purpose: To help pupils develop insight into the distributive principle for multiplication over addition, the structural basis of the usual multiplication algorithm.

Unifying Ideas: Structure; numeration; multiplication and division.

The Lesson: A few examples on the board may help pupils see a shortcut that they can use in filling in the chunks of the table:

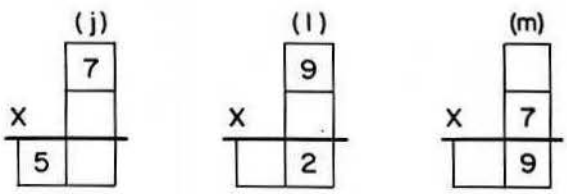
$7\times18=$___	$9\times58=$___
$7\times19=$___	$9\times59=$___
$7\times20=140$	$9\times60=540$
$7\times21=$___	$9\times61=$___
$7\times22=$___	$9\times62=$___

Point out, if necessary, that since $7\times20=140$ it follows that $7\times19=140-7=133$ and $7\times18=133-7=126$.

$8\times38=$___	$80\times28=$___
$8\times39=$___	$80\times29=$___
$8\times40=320$	$80\times30=2400$
$8\times41=$___	$80\times31=$___
$8\times42=$___	$80\times32=$___

With this introduction, pupils will see a purpose in beginning with the row of the table to the right of the 50 and the column below the 30. Now they can work more speedily both ways from those entries.

Some of the "fill in the missing digits" examples are straightforward multiplication or division problems. However, many require a bit of thinking:



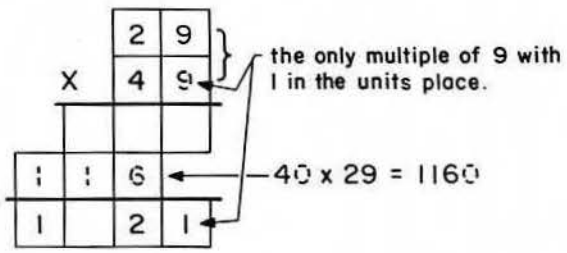
These are ways of asking three somewhat similar questions:

j. What two-digit multiple of 7 has 5 as its tens digit? There is only one: $7\times7=49$, $8\times7=56$, $9\times7=63$.

l. What two-digit multiple of 9 has a 2 in the units place? Only one: $9\times8=72$.

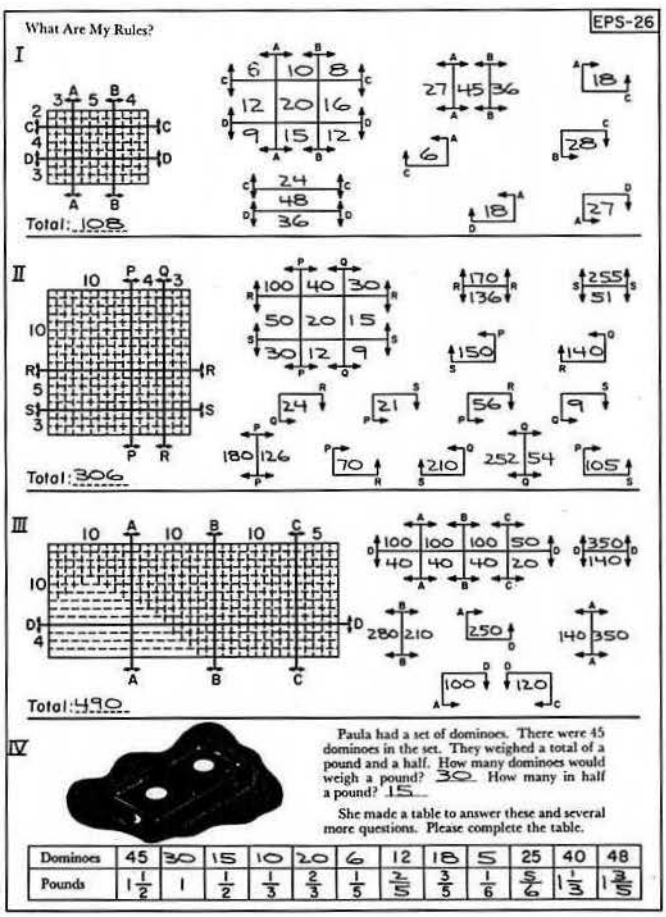
m. What two-digit multiple of 7 has a 9 in the units place? Only one: $7\times7=49$.

Similar, though often more extensive, reasoning is required of other examples. If some pupils have difficulty in getting under way, provide another clue and explain how they might have uncovered it themselves. For instance:



Either clue ought to break any stalemate.

Practice sheet 26 extends the investigation begun on practice sheet 25.



1.

X	3	4
7	21	28
8	24	32

3.

X	9	10
5	45	50
6	54	60

5.

X	5	6
8	40	48
9	45	54

2.

X	8	9
4	32	36
5	40	45

4.

X	0	1
7	0	7
8	0	8

6.

X	7	8
8	56	64
9	63	72

7.

X	7	10	17
4	28	40	68
10	70	100	170
14	98	140	238

8.

X	2	30	32
6	12	180	192
9	18	270	288
15	30	450	480

In exercises 9-11, the numbers to be written in the upper green box is the sum of the numbers in the white boxes. In exercises 12-17, each green box should contain the sum of the numbers in the white boxes to the left of it or above it.

9.

X	9	7	16
8	72	56	128

10.

X	2	11	13
7	14	77	91

11.

X	6	8	14
9	54	72	126

12.

X	8	5	13
9	72	45	117
6	48	30	78
15	120	75	195

13.

X	7	11	18
7	49	77	126
5	35	55	90
12	84	132	216

14.

X	12	13	25
6	72	78	150
8	96	104	200
14	168	182	350

15.

X	20	5	25
10	200	50	250
4	80	20	100
14	280	70	350

16.

X	10	9	19
10	100	90	190
2	20	18	38
12	120	108	228

17.

X	12	7	19
3	36	21	57
9	108	63	171
12	144	84	228

27-30 Puzzle exploring the multiplication and division algorithms

Tasks: *Pupils reconstruct the multiplication grids.*

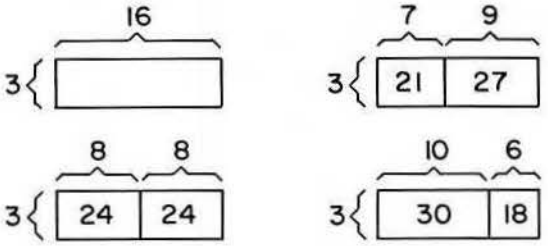
Purpose: To help pupils develop insight into the distributive principle for multiplication over addition, the structural basis of the usual multiplication and division algorithms.

Unifying Ideas: Structure; numeration; multiplication and division.

The Lesson: The problems presented on pupil pages 27 and 28 (and practice sheets 27 and 28) essentially involve the lines of thought required in "long division"—undoing a "long multiplication."

Teachers are so familiar with the usual algorithms for multiplication and division that they sometimes fail to help pupils understand that the basic strategy behind these algorithms has nothing to do with place value. Rather, the basic strategy of these algorithms is to break up a problem into manageable pieces and then to assemble the results. Planning to utilize the bookkeeping advantages of place value is, in fact, a kind of systematization of the more fundamental notion.

$$\begin{array}{r} 16 \\ \times\ 3 \\ \hline 48 \end{array} \qquad \begin{array}{r} 7+9 \\ \times\ 3 \\ \hline 27 \\ 21 \\ \hline 48 \end{array} \qquad \begin{array}{r} 8+8 \\ \times\ 3 \\ \hline 24 \\ 24 \\ \hline 48 \end{array} \qquad \begin{array}{r} 10+6 \\ \times\ 3 \\ \hline 18 \\ 30 \\ \hline 48 \end{array}$$



Similarly, "$48 \div 3 = ?$" can be broken up in several ways, such as:

$$3\overline{)\,21 + 27}\ \ 7 + 9 = 16 \qquad 3\overline{)\,24 + 24}\ \ 8 + 8 = 16$$

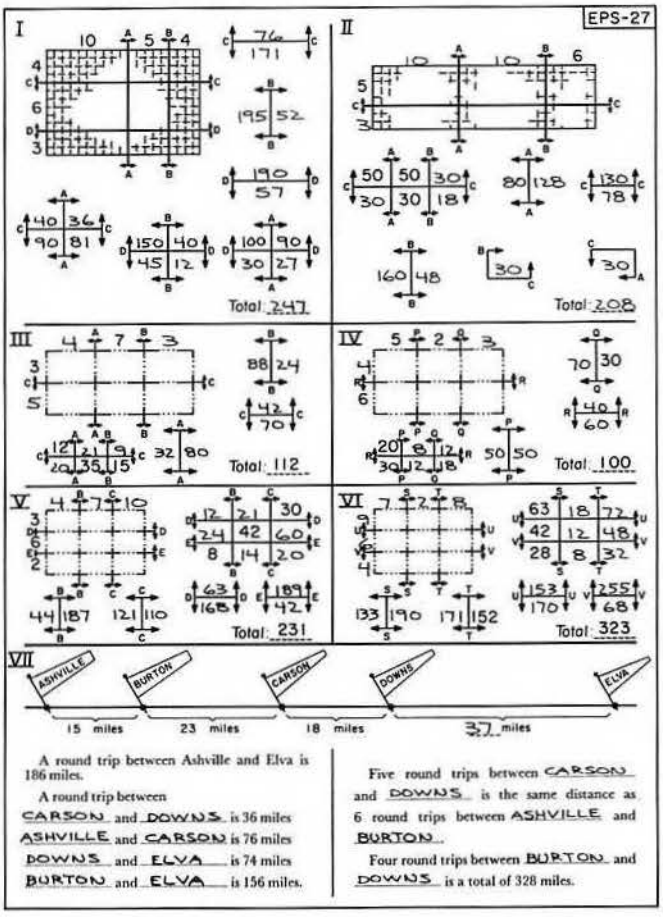
Division usually provides information about a product—the result of a multiplication. We know that one of the factors can be found if the other factor is given. For example,

$$12\overline{)\,216}$$

may be understood as asking how many 12-stone piles could be made from a pile of 216 stones. If we know that $12 \times 12 = 144$, we know that we could put 12 in each pile and have $216 - 144$, or 72, left to distribute. Since $6 \times 12 = 72$, we could put 6 more in each pile, making a total of 18 in each pile. Thus, $216 \div 12 = 18$.

After we understand the basic notion of reconstructing a multiplication, we can look for the shortcuts that attention to place value usually provide. Then we would break the problems up in some such way as:

$$\begin{array}{rl} 10 \times 12 = 120 & \qquad (216 - 120 = 96) \\ 8 \times 12 = 96 & \\ \hline 18 \times 12 = 216 & \end{array}$$



Rule I: Same as bottom of the last page: the number to be written in a green box is the sum of the numbers in the white boxes to the left of it or above it.

1.

X	9	6	15
5	45	30	75

2.

X	10	3	13
7	70	21	91

3.

X	30	8	38
4	120	32	152

Rule II: In any box that is checked (√), write the largest multiple of ten you can and still be able to follow Rule I. The second and third examples above follow Rule II, but the first does not.

4.

X	√ 10	2	12
7	70	14	84

5.

X	√ 50	8	58
5	250	40	290

6.

X	√ 40	6	46
8	320	48	368

7.

X	√ 60	8	68
9	540	72	612

8.

X	√ 70	7	77
6	420	42	462

9.

X	√ 10	2	12
12	120	24	144

10.

X	√ 0	9	9
12	0	108	108

11.

X	√ 20	5	25
15	300	75	375

12.

X	√ 10	8	18
14	140	112	252

Rule II must be applied twice in each of the following:

13.

	√ 20	3	23
√ 10	200	30	230
3	60	9	69
13	260	39	299

14.

	√ 30	5	35
√ 20	600	100	700
6	180	30	210
26	780	130	910

15.

	√ 10	9	19
√ 40	400	360	760
6	60	54	114
46	460	414	874

16.

	√ 60	3	63
√ 20	1,200	60	1,260
7	420	21	441
27	1,620	81	1,701

17.

	√ 100	3	103
√ 10	1,000	30	1,030
8	800	24	824
18	1,800	54	1,854

18.

	√ 50	7	57
√ 50	2,500	350	2,850
7	350	49	399
57	2,850	399	3,249

Tasks: *Pupils reconstruct the multiplication grids.*

Purpose: To help pupils develop insight into the distributive principle for multiplication over addition, the structural basis of the usual multiplication and division algorithms.

Unifying Ideas: Structure; numeration; multiplication and division.

The Lesson: Suppose that a supply of stamps in the following denominations is available in an old-fashioned shipping room:

List A	1¢	2¢	3¢	4¢	5¢	6¢	7¢	8¢	9¢
List B	10¢	20¢	30¢	40¢	50¢	60¢	70¢	80¢	90¢

With experience, the shipping clerks would find that any amount of postage less than \$1.00 could be made without using more than one stamp from each list. If they shipped four packages, each using the same amount of postage, and if the total amount of postage was \$1.52, we might keep track this way:

No. of Pkg.	B	A	
	?	?	? Postage on each
4	\$?	\$?	\$1.52 Total postage

How does one think about the problem of filling in the missing information? Perhaps we might begin by wondering what stamp, if any, was used from list A. After a little thought, we can limit the possibilities by noticing that in 152 the units place is occupied by a 2. Since $4 \times 3 = 12$ and $4 \times 8 = 32$, we could conclude that the stamp from list A must be a 3¢ or an 8¢ stamp.

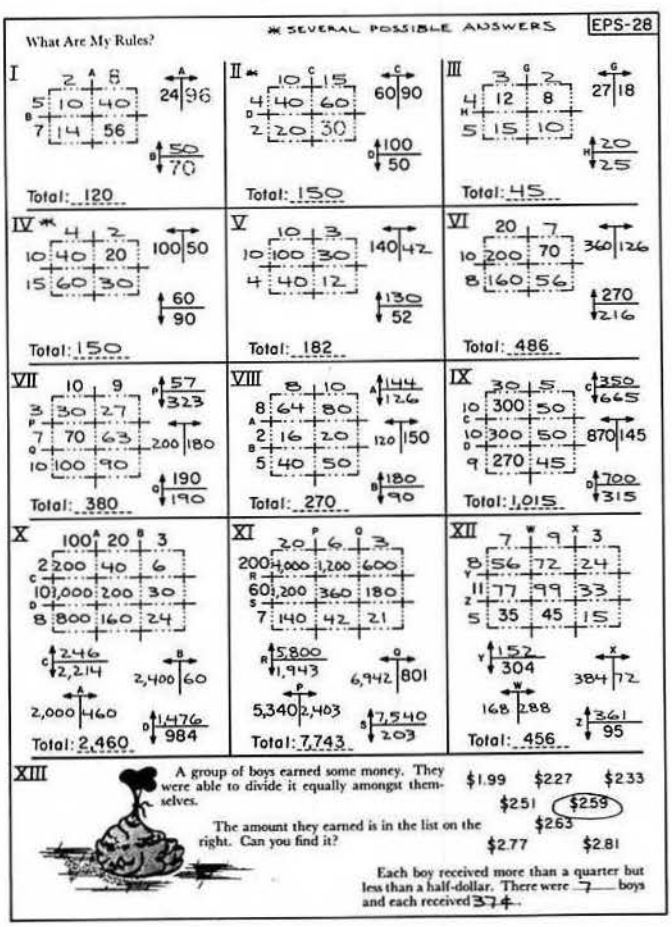
That's not enough information. So, let's wonder what stamp from list B was on each of the four packages with a total postage of \$1.52. Could it have been 10¢, 20¢, 30¢, 40¢, 50¢? Certainly not 50¢, because four of these stamps would total \$2.00, which is more than \$1.52. Nor could it have been 40¢, since $40 \times 4 = 160$. It could, however, have been a 30¢ stamp on each, accounting for \$1.20 of the total and leaving 32¢ to be accounted for. So the stamp from list A would have been an 8¢ stamp—$8¢ \times 4 = 32¢$; and each package had $30¢ + 8¢$, or 38¢ postage. We can complete the record:

No. of Pkg.	B	A	
	30¢	8¢	38¢ Postage on each
4	\$1.20	\$.32	\$1.52 Total postage

Several examples such as this will prepare the pupils for pupil page 28 and practice sheet 28.

More important, these activities include precisely the kind of thinking we use when tackling a division problem. That is, selecting the largest stamp from list B is the same strategy we use in finding a "trial divisor," and the rest of the work has its exact counterpart in the division algorithm.

The purpose of these activities is to keep the pupils thinking as they move toward a procedure that eventually becomes automatic. Otherwise, the rules come before understanding—and this usually precludes developing any motivation for understanding.



a.

x	9
4	36

b.

x	18
2	36

c.

x	6
9	54

d.

x	5
8	40

e.

x	7
9	63

f.

x	12
10	120

g.

x	12
8	96

h.

x	21
6	126

i.

	20
5	100

j.

	40
5	200

k.

	8
8	64

l.

	11
9	99

m. $7\overline{)56}$ = 8

n. $9\overline{)72}$ = 8

o. $4\overline{)100}$ = 25

p. $2\overline{)130}$ = 65

q. $3\overline{)150}$ = 50

r. $10\overline{)130}$ = 13

s. $12\overline{)120}$ = 10

t. $25\overline{)300}$ = 12

u. $50\overline{)1{,}000}$ = 20

v. $30\overline{)600}$ = 20

w. $20\overline{)420}$ = 21

x. $20\overline{)500}$ = 25

x	9
5	45

$5\overline{)45}$ = 9 ..... in shorthand

Use any correct method you know of to complete the following entries in the large chart. Look for shortcuts.

1. $8\overline{)72}$ = 9
2. $4\overline{)72}$ = 18
3. $25\overline{)800}$ = 32
4. $50\overline{)800}$ = 16
5. $70\overline{)910}$ = 13
6. $35\overline{)910}$ = 26
7. $40\overline{)1{,}200}$ = 30
8. $80\overline{)1{,}200}$ = 15
9. $40\overline{)1{,}320}$ = 33
10. $20\overline{)1{,}320}$ = 66
11. $43\overline{)903}$ = 21
12. $43\overline{)1{,}806}$ = 42

13. $26\overline{)78}$ = 3 → 14. $26\overline{)520}$ = 20 → 15. $26\overline{)598}$ = 23 → 16. $13\overline{)598}$ = 46 → 17. $13\overline{)1{,}196}$ = 92 → 18. $26\overline{)1{,}196}$ = 46

19. $72\overline{)360}$ = 5 → 20. $72\overline{)2{,}160}$ = 30 → 21. $72\overline{)2{,}520}$ = 35 → 22. $36\overline{)1{,}260}$ = 35 → 23. $18\overline{)1{,}260}$ = 70 → 24. $18\overline{)2{,}520}$ = 140

25. $35\overline{)700}$ = 20 → 26. $35\overline{)140}$ = 4 → 27. $35\overline{)840}$ = 24 → 28. $79\overline{)474}$ = 6 → 29. $79\overline{)1{,}580}$ = 20 → 30. $79\overline{)2{,}054}$ = 26

31. $13\overline{)520}$ = 40 → 32. $13\overline{)104}$ = 8 → 33. $13\overline{)624}$ = 48 → 34. $26\overline{)624}$ = 24 → 35. $39\overline{)624}$ = 16 → 36. $52\overline{)624}$ = 12

Tasks: Pupils complete the multiplication and division problems.

Purpose: To help pupils develop insight into the distributive principle for multiplication over addition, the structural basis of the usual multiplication and division algorithms.

Unifying Ideas: Structure; numeration; multiplication and division.

The Lesson: Pupil page 29, in the first few examples, continues the "hunt in the table," but then fades into a shorthand that is the usual "division form":

×	
4	36

→ 7 ⟌ 56 → 8 ⟌ 72

However, to help underline the idea that division is "undoing a multiplication," we include a version that implies multiplication. The form:

$$10 \overline{)\ 12}$$

asks for the product of 10 and 12. This variation:

$$\overline{)72}\ 8$$

is a slightly unorthodox way to ask for a missing factor. Such variations encourage maximum understanding as pupils carry out computations.

Exercises 1-36 of pupil page 29 are arranged to shed more light on division and multiplication. Blue arrows indicate pairs or sequences of exercises that are linked to each other. Consider exercises 1 and 2:

$$8 \overline{)72}\ 9 \quad \longleftrightarrow \quad 4 \overline{)72}$$

If there are 9 groups of 8 in 72, how many groups of 4 are there? Answer: twice as many, or 18. Consider exercises 11 and 12:

$$43 \overline{)903}\ 21 \quad \longleftrightarrow \quad 43 \overline{)1806}$$

If there are 21 groups of 43 in 903, how many groups of 43 are there in 1,806 (which is twice as many as 903)? Answer: twice as many, or 42. Consider exercises 25, 26, and 27:

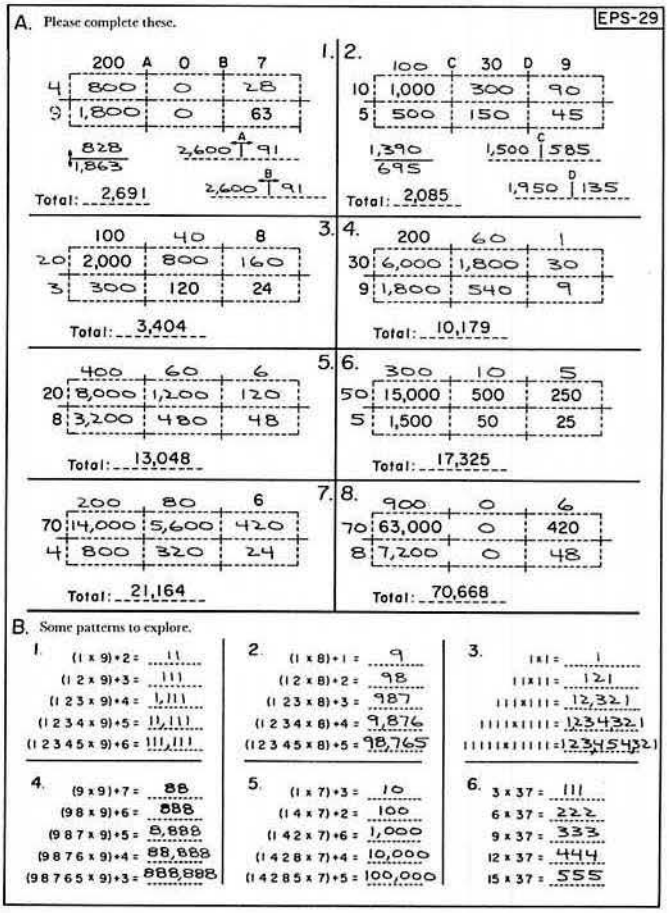
$$35 \overline{)700}\ 20 \rightarrow 35 \overline{)140}\ 4 \rightarrow 35 \overline{)840}$$

If there are 20 groups of 35 in 700 and 4 groups of 35 in 1440, and if 840 is 700 + 140, how many groups of 35 are there in 840? Answer: 20 + 4, or 24.

Inability to follow this kind of thinking may be concealed by mastery of the algorithm:

$$35 \overline{)840}\ 24$$
$$70$$
$$140$$
$$140$$

While we wish pupils to know and use this algorithm, too much emphasis on its formal aspects is likely to result in their getting the right answers without having the faintest notion of why the rules work. So, when the rules fade a bit in the memory and become confused with other rules or when slight variations in the questions arise, such pupils may be helpless. We must, therefore, continue to explore ways in which rules are backed up with understanding.



I.

X	7	10	17	100	117
3	21	30	51	300	351
6	42	60	102	600	702
9	63	90	153	900	1,053
27	189	270	459	2,700	3,159

Skipping Around . . . in the table . . . and beyond the table.

LOOK FOR SHORTCUTS!

You may be able to find all the entries without even using scratch paper.

II.

X	5	20	25	100	125
4	20	80	100	400	500
12	60	240	300	1,200	1,500
24	120	480	600	2,400	3,000
48	240	960	1,200	4,800	6,000
72	360	1,440	1,800	7,200	9,000

III.

X	8	30	38	200	238
4	32	120	152	800	952
16	128	480	608	3,200	3,808
20	160	600	760	4,000	4,760
36	288	1,080	1,368	7,200	8,568
18	144	540	684	3,600	4,284

IV.

	300	10	9	319
8	2,400	80	72	2,552
16	4,800	160	144	5,104
24	7,200	240	216	7,656
40	12,000	400	360	12,760

V.

	100	30	8	138
6	600	180	48	828
70	7,000	2,100	560	9,660
100	10,000	3,000	800	13,800
176	17,600	5,280	1,408	24,288

The arrows are hints to shortcuts you may want to use.

a.

X	30
14	420

b. 14 | 98 → 7

c. 14 | 518 → 37

d. 28 | 5,180 → 185

e. 34 | 3,400 → 100

f. 28 | 420 → 15

g. 14 | 196 → 14

h. 7 | 196 → 28

i. 7 | 392 → 56

j. 34 | 1,700 → 50

k. 14 | 840 → 60

l. 14 | 392 → 28

m. 14 | 1,232 → 88

n. 7 | 784 → 112

o. 34 | 272 → 8

p. 42 | 840 → 20

q. 28 | 784 → 28

r. 28 | 1,568 → 56

s. 14 | 1,568 → 112

t. 34 | 5,372 → 158

Tasks: *Pupils reconstruct multiplication grids and complete multiplication-division problems.*

Purpose: To help pupils develop insight into the distributive principle for multiplication over addition, the structural basis of the usual multiplication and division algorithms.

Unifying Ideas: Structure; numeration; multiplication and division.

The Lesson: Pupil page 30 concludes this direct examination and development of multiplication and division algorithms. However, we do not expect full mastery, particularly of the division algorithm. Pupil pages 36 through 49 present an intensified study of the basis for and development of the usual division algorithm.

Exercises I-V require "undoing a multiplication" —finding a missing factor when the product and one factor are given. That is what we mean by division. After finding the missing numbers at the top of a grid, you can find the rest of the entries by straightforward multiplication.

The shortcuts mentioned follow from relationships among the numbers selected for the green blocks. Consider exercise I:

		7	10	(7+10) 17	100	(17+100) 117
	3	21	30	51	300	351
(twice 3)	6	42				
(3+6)	9	63				
(3 times 9)	27	189				

3 x 7 = 21
6 x 7 = 2 x 21 = 42
9 x 7 = 21 + 42 = 63
27 x 7 = 3 x 63 = 189

3 x 7 = 21
3 x 10 = 30
3 x 17 = 21 + 30 = 51
3 x 100 = 300
3 x 117 = 51 + 300 = 351

Similar relationships appear in exercises II-V.

Interrelationships among the exercises at the bottom of the page will lead pupils who understand the fundamental idea of division (and multiplication) to shortcuts. Consider exercises *a-d*:

$$14\overline{)420} \leftrightarrow 14\overline{)98} \leftrightarrow 14\overline{)518} \leftrightarrow 28\overline{)5180}$$

If you notice that $518 = 420 + 98$, then you know from the first two exercises that $518 \div 14 = 30 + 7$. The next step requires a two-step bit of thinking. Notice that the given product (or dividend), 5180, is 518×10; so, the missing factor should be 10 times as much, or 370; however, the given factor (or divisor) is twice as large; so, the result will be only half of 370, or 185; or it will be 37×5 instead of 37×10. In either case, 185.

Practice sheet 30 is an excursion into that part of the large multiplication table that ought to have been committed to memory, and just a little beyond. Question I asks for a list of prime numbers less than 100. Question II asks for a list of all composite numbers less than 100 that can be factored in only one way (excluding 1 as a factor). Question III asks for all composite numbers less than 85 that can be factored in more than 1 way (excluding 1 as a factor).

Note that all questions are concerned only with entries in the table on or below the diagonal of square numbers. There are, of course, no entries above the diagonal in the table as we have designed it.

EPS-30

I. List all numbers larger than 1 and less than 100 that appear only once in the multiplication table (below the diagonal of square numbers).

2 3 5 7 11 13 17 19 23 29
31 37 41 43 47 53 59 61 67 71
73 79 83 89 97

II. List all numbers larger than 1 and less than 100 that appear exactly twice in the multiplication table (on or below the diagonal of square numbers).

4 6 8 9 10 14 15 21 22 25
26 27 33 34 35 38 39 46 49 51
55 57 58 62 65 69 74 77 82 85
86 87 91 93 94 95

III. Given below are all the numbers larger than 1 and less than 84 that appear at least three times in the multiplication table (on or below the diagonal of square numbers). List the numbers of the columns in which the given numbers appear.



12: 1 2 3
16: 1 2 4
18: 1 2 3
20: 1 2 4
24: 1 2 3 4
28: 1 2 4
30: 1 2 3 5
32: 1 2 4
36: 1 2 3 4 6
40: 1 2 4 5
42: 1 2 3 6
44: 1 2 4
45: 1 3 5
48: 1 2 3 4 6
50: 1 2 5
52: 1 2 4
54: 1 2 3 6
56: 1 2 4 7
60: 1 2 3 4 5 6
63: 1 3 7
64: 1 2 4 8
66: 1 2 3 6
68: 1 2 4
70: 1 2 5 7
72: 1 2 3 4 6 8
75: 1 3 5
76: 1 3 7
78: 1 2 3 6
80: 1 2 4 5 8
81: 1 3 9

What's My Rule?

Call out a number. I'll write it down. Then I'll use my rule and write a number under it. When you discover my rule, you can fill in the blanks in each record.

a.

If you call out	7	15	0	11	29	153
then I'll write	10	18	3	14	32	156

b.

	8	3	10	26	57	0	169	
	16	6	20	52	114	0	338	

c.

	6	10	5	9	0	15	150	
	19	31	16	28	1	46	451	

d.

	4	9	7	5	8	6	15	
	16	81	49	25	64	36	225	

Now, I'll use an n, an a, or an x or whatever letter comes to my head to indicate the number you call out. I will include "pattern indicators" as clues to my rules.

1.

n	4	10	34	83	16	
n + 17	21	27	51	100	33	

2.

a	10	4	15	27	118	
2a − 3	17	5	27	51	233	

3.

x	3	11	10	8	40	
$x^2 + 1$	10	122	101	65	1,601	

4.

m	34	5	43	50	22	
50 − m	16	45	7	0	28	

What Are My Rules?

Call out a number. I'll write it down. Then I'll use several rules and note the results. When you discover my rules you can use them to fill in the blanks in each record.

I.

x	6	10	8	15	0
3x + 2x	30	50	40	75	0
$x^2 + 1$	37	101	65	226	1
5x	30	50	40	75	0

II.

a	6	10	21	16	50
a − 1	5	9	20	15	49
a · (a − 1)	30	90	420	240	2,450
$a^2 - a$	30	90	420	240	2,450

III.

n	5	3	8	11	20
5 · (n − 1)	20	10	35	50	95
3 · (n + 2)	21	15	30	39	66
$n^2 - n$	20	6	56	110	380
3n + 6	21	15	30	39	66
n · (n − 1)	20	6	56	110	380
5n − 5	20	10	35	50	95
6n ÷ 3	10	6	16	22	40

IV.

x	4	13	7	10
x + 3	7	16	10	13
x − 3	1	10	4	7
(x + 3) · (x − 3)	7	160	40	91
x^2	16	169	49	100
$x^2 - 9$	7	160	40	91
$x^2 + 6x + 9$	49	256	100	169
$(x + 3)^2$	49	256	100	169

Tasks: *Pupils complete all tables.*

Purpose: To help pupils think about and discover patterns in arithmetic and understand the algebraic language that expresses these patterns.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: We are concerned on pupil pages 31, 32, and 33 with a deeper investigation of some patterns that have been noticed along the way. In our study we shall borrow a bit of shorthand from algebra to help talk about these patterns in an economical way.

The letters in algebraic expression should come in quite naturally and need little discussion. We discourage any reference to them as "unknowns" or "variables" or "literal numbers." If pupils need a name for these letters, you might call them "place holders" since they point out places for numbers in the pattern indicators under consideration.

If n is the number called out in examples *a-d* of *What's My Rule?* then the rules can be expressed by these pattern indicators:

a. $n+3$: add three to the number called out;
b. $2\times n$ or $2n$: double the number;
c. $3n+1$: triple the number and add 1;
d. $n\times n$ or n^2: multiply the number by itself.

In these examples, some pattern in the two or three episodes recorded is a clue to the rule:

$$7 \rightarrow 10$$
$$15 \rightarrow 18$$
$$0 \rightarrow 3$$
$$n \rightarrow n+3$$

In examples 1-4 at least one entry is reported in each. In these examples pupils need to rely more on the pattern indicators than in earlier examples on the page.

In examples I-IV the pattern indicators are indispensable, since only one entry is given in each column. Certain rows of entries are identical in each, and would be regardless of the number called out. In I the expressions:

$$3x + 2x \quad \text{and:} \quad 5x$$

seem to describe two different ways to reach identical results regardless of the number called out. So, we know what is meant by saying that, for each number x,

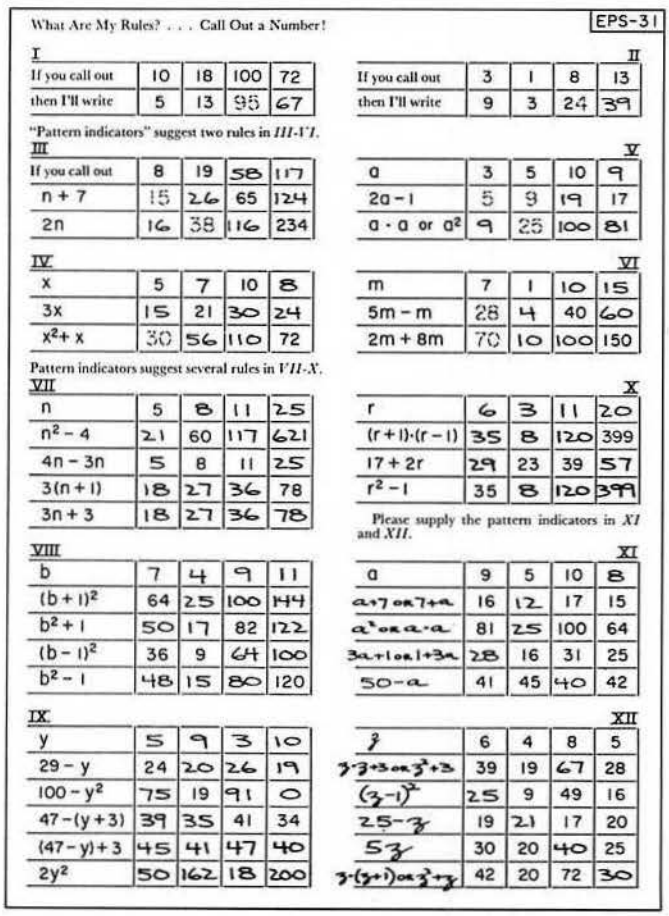
$$3x + 2x = 5x$$

Similarly, when identical rows turn up in the other examples, we conclude that:

$$a \times (a-1) = a^2 - a$$
$$5 \times (n-1) = 5n - 5$$
$$3 \times (n+2) = 3n + 6$$
$$n^2 - n = n \times (n-1)$$
$$(x+3) \times (x-3) = x^2 - 9$$
$$x^2 + 6x + 9 = (x+3)^2$$

notice similarity of ideas

Practice sheet 31 provides other similar examples in which identical rows turn up, leading us to conclude that one rule produces the same results as another.



What Are My Rules? . . . Call Out a Number!

EPS-31

I

If you call out	10	18	100	72
then I'll write	5	13	95	67

II

If you call out	3	1	8	13
then I'll write	9	3	24	39

"Pattern indicators" suggest two rules in *III-VI*.

III

If you call out	8	19	58	117
n + 7	15	26	65	124
2n	16	38	116	234

V

a	3	5	10	9
2a - 1	5	9	19	17
a · a or a^2	9	25	100	81

IV

x	5	7	10	8
3x	15	21	30	24
$x^2 + x$	30	56	110	72

VI

m	7	1	10	15
5m - m	28	4	40	60
2m + 8m	70	10	100	150

Pattern indicators suggest several rules in *VII-X*.

VII

n	5	8	11	25
$n^2 - 4$	21	60	117	621
4n - 3n	5	8	11	25
3(n + 1)	18	27	36	78
3n + 3	18	27	36	78

X

r	6	3	11	20
(r + 1)·(r - 1)	35	8	120	399
17 + 2r	29	23	39	57
$r^2 - 1$	35	8	120	399

Please supply the pattern indicators in *XI* and *XII*.

VIII

b	7	4	9	11
$(b+1)^2$	64	25	100	144
$b^2 + 1$	50	17	82	122
$(b-1)^2$	36	9	64	100
$b^2 - 1$	48	15	80	120

XI

a	9	5	10	8
a+7 or 7+a	16	12	17	15
a^2 or a·a	81	25	100	64
3a+1 or 1+3a	28	16	31	25
50 - a	41	45	40	42

IX

y	5	9	3	10
29 - y	24	20	26	19
$100 - y^2$	75	19	91	0
47 - (y + 3)	39	35	41	34
(47 - y) + 3	45	41	47	40
$2y^2$	50	162	18	200

XII

z	6	4	8	5
3·z+3 or $3z^2+3$	39	19	67	28
$(z-1)^2$	25	9	49	16
25 - z	19	21	17	20
5z	30	20	40	25
3·(z+1) or $3z^2+z$	42	20	72	30

Call out a pair of numbers. I'll use one or more rules and note some of the results. I'll use "pattern indicators" to show my rules. Please complete each record.

1.

a	7	10	9	23	66
b	3	8	9	18	19
a − b	4	2	0	5	47
a + b	10	18	18	41	85

2.

x	9	12	20	5	9
y	3	5	9	5	7
xy	27	60	180	25	63
x − y	6	7	11	0	2

3.

m	8	10	8	12	15
n	5	4	8	7	19
m + n	13	14	16	19	34
2m + n	21	24	24	31	49
m + 2n	18	18	24	26	53

4.

f	5	4	9	10	14
r	2	4	8	6	11
$f^2 + r^2$	29	32	145	136	317
$(f + r)^2$	49	64	289	256	625
$(f - r)^2$	9	0	1	16	9

5.

c	9	13	18	8	11
d	4	6	10	8	7
c − d	5	7	8	0	4
c + 2d	17	25	38	24	25
c · (d + 2)	54	104	216	80	99
2c + 3d	30	44	66	40	43
cd	36	78	180	64	77
2cd − 1	71	155	359	127	153

6. (make up your own)

a	7				
b	3				
2a − b	11				
2ab	42				
2(a − b)	8				
$(a - b)^2$	16				
$a^2 - b^2$	40				
$a^2 + b^2$	58				

Notice the green-tinted rows below. Perhaps you would like to try other numbers for *a* and *b* and for *s* and *t*. What are your conclusions?

7. $(a-b)^2 = a^2 - 2ab + b^2$

a	8	10	9		
b	3	4	7		
a − b	5	6	2		
$(a - b)^2$	25	36	4		
a^2	64	100	81		
b^2	9	16	49		
2ab	48	80	126		
$a^2 - 2ab + b^2$	25	36	4		

8. $(s+t)^2 = s^2 + 2st + t^2$

s	5	13		
t	4	2		
s + t	9	15		
$(s + t)^2$	81	225		
s^2	25	169		
t^2	16	4		
2st	40	52		
$s^2 + 2st + t^2$	81	225		

Tasks: *Pupils complete all tables. They choose their own numbers in exercises 6, 7, and 8.*

Purpose: To help pupils think about and discover patterns in arithmetic and understand the algebraic language that expresses these patterns.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: A new twist is added on pupil page 32. It seems at first that some of the examples have insufficient information. For example, the last column of the first example says, in effect: "Find two numbers whose difference is 47 and whose sum is 85."

Random trial and error will be very tedious but educated trial and error will lead to the solution without much delay. Try 50 and 35—the difference is too small. So try 60 and 25—the difference is still too small, but we're getting closer. 65 and 20 have a difference of 45. So, 66 and 19 have 85 as their sum and 45+2, or 47, as their difference. Some pupils will develop an educated trial-and-error system as they work.

Exercises 1-6 are designed so that pupils may see the very different results of rules whose expressions are similar:

$2m + n$ and $m + 2n$

$f^2 + z^2$ and $(f + z)^2$

$a^2 - b^2$ and $(a - b)^2$

$2a - b$ and $2(a - b)$

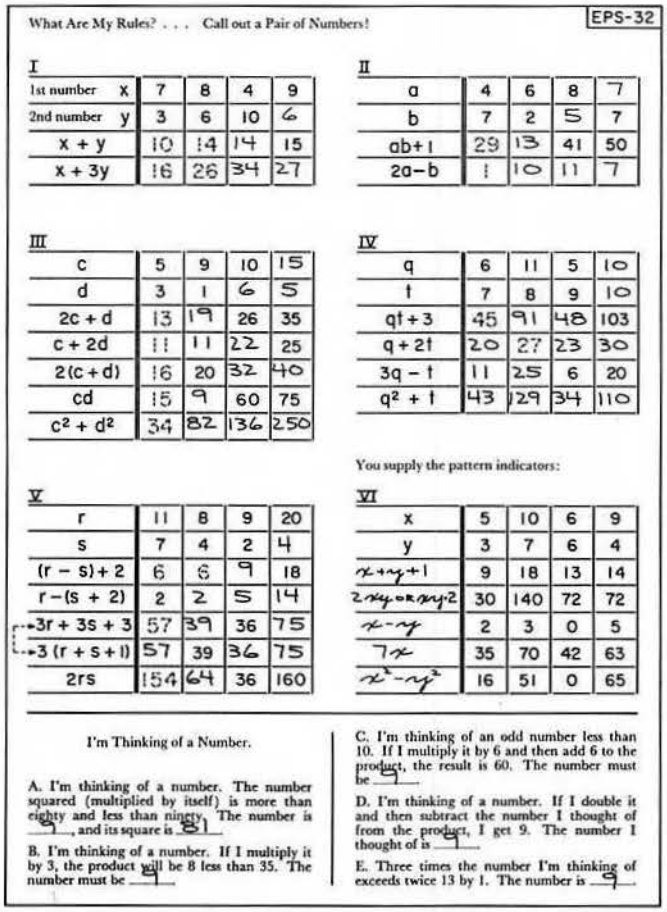
Students are urged to make up their own examples in exercises 6-8.

As in many instances on the previous page, exercises 7 and 8 each have two identical lines of results. These lead to two rather unexpected conclusions. For all numbers a and b, and for all numbers s and t,

$$(a - b)^2 = a^2 - 2ab + b^2 \text{ and}$$
$$(s + t)^2 = s^2 + 2st + t^2.$$

Pupils might well compare notes to see if anyone selected pairs of numbers for which those conclusions do not hold. Perhaps someone with more experience might investigate the results of selecting pairs of fractions. Others might try very large numbers. Others may try pairs in which one or both of the numbers is zero—searching for a counterexample that would, of course, lead us to scrap the conclusions. None will develop unless someone does some bad arithmetic. For the results in exercises 6 and 7 to make sense to a pupil who has not studied negative numbers, a and b must be so chosen that b is not greater than a.

Practice sheet 32 extends the activity, bringing in two variations. In exercise VI pupils determine their own pattern indicators. Exercises A-E present pattern-indicator problems in verbal form; some pupils might like to write out pattern indicators for these examples. In both of these variations, accept any pattern indicator that expresses a correct rule. For example, $2xy$, $2 \times x \times y$, $2yx$, and twice $x \times y$ are equivalent to each other, and each could correctly serve as a pattern indicator for the fourth row in exercise VI.



What Are My Rules? . . . Call out a Pair of Numbers! EPS-32

I

1st number x	7	8	4	9
2nd number y	3	6	10	6
x + y	10	14	14	15
x + 3y	16	26	34	27

II

a	4	6	8	7
b	7	2	5	7
ab+1	29	13	41	50
2a−b	1	10	11	7

III

c	5	9	10	15
d	3	1	6	5
2c + d	13	19	26	35
c + 2d	11	11	22	25
2(c + d)	16	20	32	40
cd	15	9	60	75
$c^2 + d^2$	34	82	136	250

IV

q	6	11	5	10
t	7	8	9	10
qt+3	45	91	48	103
q+2t	20	27	23	30
3q − t	11	25	6	20
$q^2 + t$	43	129	34	110

You supply the pattern indicators:

V

r	11	8	9	20
s	7	4	2	4
(r − s)+2	6	6	9	18
r−(s + 2)	2	2	5	14
3r + 3s + 3	57	39	36	75
3 (r + s + 1)	57	39	36	75
2rs	154	64	36	160

VI

x	5	10	6	9
y	3	7	6	4
x+y+1	9	18	13	14
2xy or xy2	30	140	72	72
x−y	2	3	0	5
7x	35	70	42	63
$x^2 - y^2$	16	51	0	65

I'm Thinking of a Number.

A. I'm thinking of a number. The number squared (multiplied by itself) is more than eighty and less than ninety. The number is 9, and its square is 81.

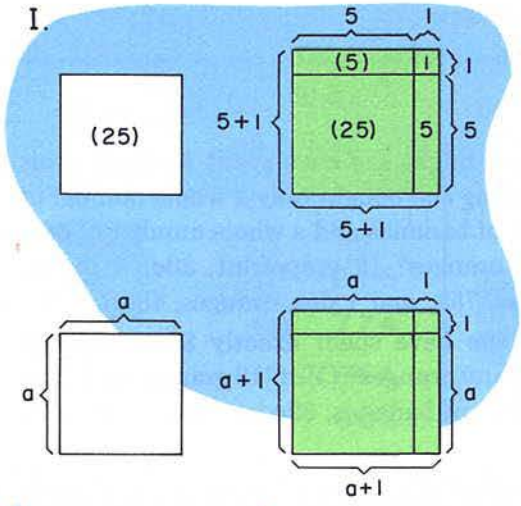
B. I'm thinking of a number. If I multiply it by 3, the product will be 8 less than 35. The number must be 9.

C. I'm thinking of an odd number less than 10. If I multiply it by 6 and then add 6 to the product, the result is 60. The number must be 9.

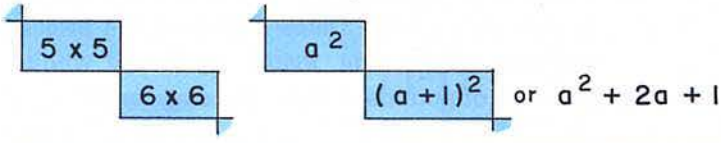
D. I'm thinking of a number. If I double it and then subtract the number I thought of from the product, I get 9. The number I thought of is 9.

E. Three times the number I'm thinking of exceeds twice 13 by 1. The number is 9.

I.

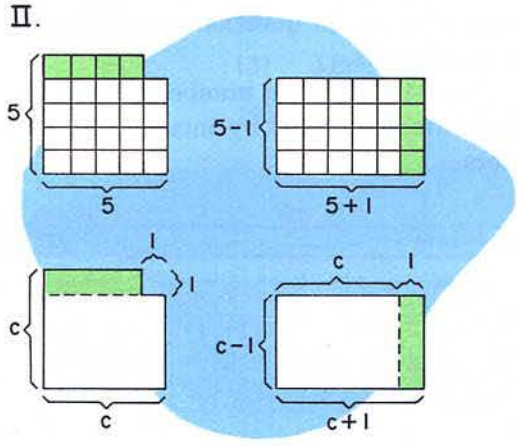


a	5	10	8	14	19	25	40
a + 1	6	11	9	15	20	26	41
$(a+1)^2$	36	121	81	225	400	676	1,681
a^2	25	100	64	196	361	625	1,600
2a + 1	11	21	17	29	39	51	81
a^2+2a+1	36	121	81	225	400	676	1,681

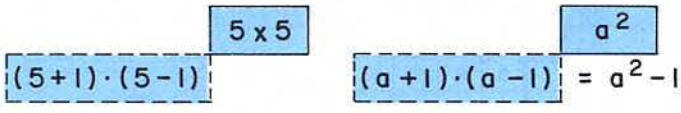


Glance back at pages 10 and 11. Moving from a^2 to $(a+1)^2$ is "moving one step down the diagonal."

II.

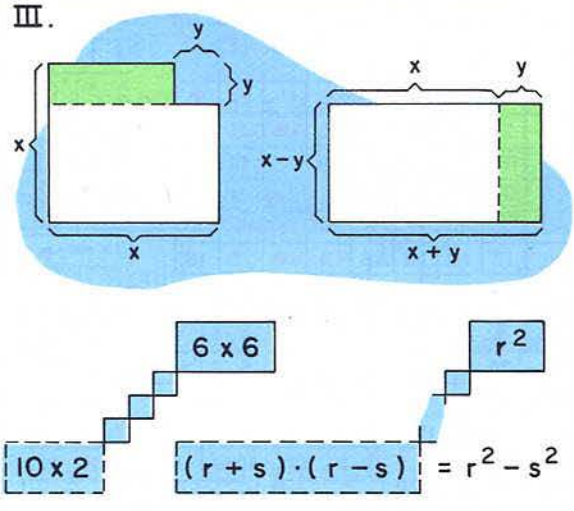


a	5	8	10	15	25	26
a^2	25	64	100	225	625	676
a^2-1	24	63	99	224	624	675
a + 1	6	9	11	16	26	27
a − 1	4	7	9	14	24	25
$(a+1)\cdot(a-1)$	24	63	99	224	624	675



Glance back at pages 12 and 13 . . . 1 step down and left.

III.



r	6	8	12	20	65
s	4	3	5	9	30
r^2	36	64	144	400	4,225
s^2	16	9	25	81	900
r^2-s^2	20	55	119	319	3,325
r + s	10	11	17	29	95
r − s	2	5	7	11	35
$(r+s)\cdot(r-s)$	20	55	119	319	3,325

Glance again at pages 12 and 13 . . . *s* steps down and left.

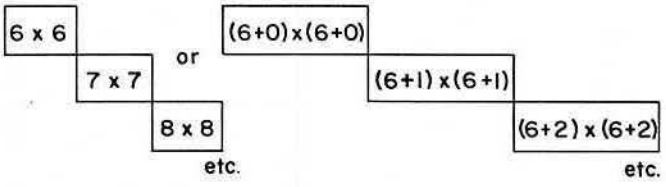
33 Exploring the patterns (a+1)^2 = a^2 + 2a + 1; (a+1) (a-1) = a^2 - 1; and (r+s) (r-s) = r^2 - s^2

Tasks: *Pupils complete all tables.*

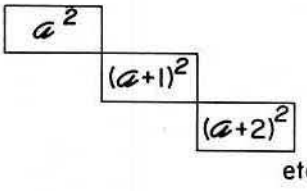
Purpose: To help pupils think about and discover patterns in arithmetic and understand the algebra which expresses these patterns. To help pupils relate the patterns to graphic representations.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Our study of ways to use pattern indicators leads to a second look at patterns that developed in working out the multiplication table earlier in the book. First, in exercise I we look at the square numbers in the diagonal, 1, 4, 9, 16, etc. These entries could be written:



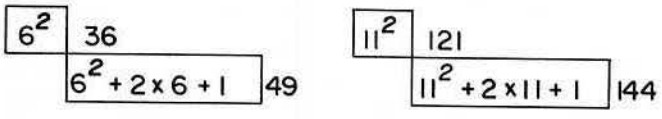
or, in a more general way:



Our first experiment leads us to the conclusion that, for each number a,

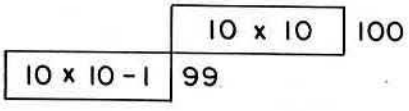
$$(a+1)^2 = a^2 + 2a + 1.$$

Let's test that:

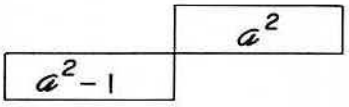


So we could move from any entry on that diagonal to the next larger entry by adding 1 more than twice the column number of the smaller entry.

In exercise II we investigate the first entry down and left from entries in the diagonal of square numbers. Easy. We simply subtract 1:



or, generally:

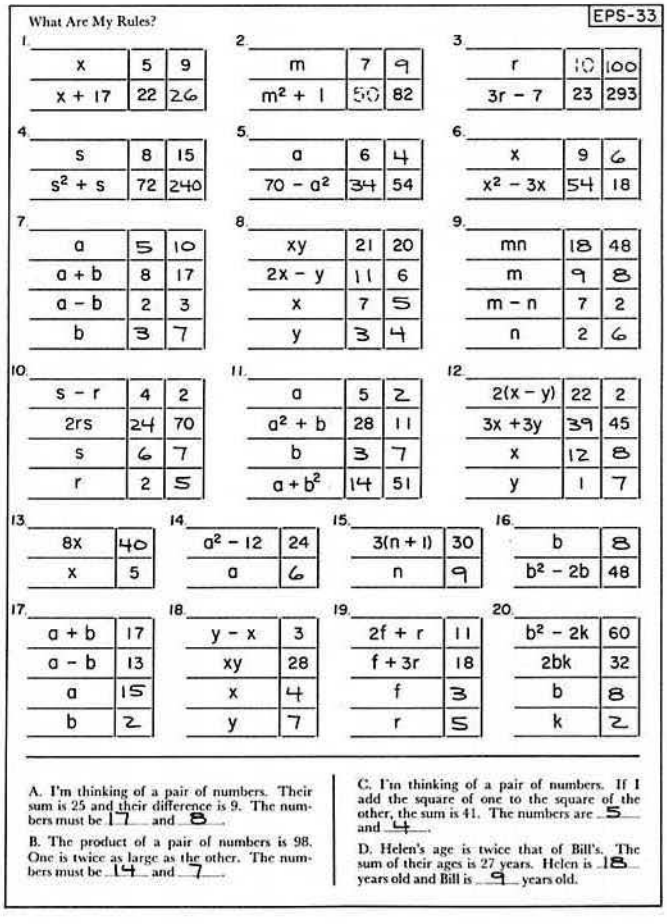


Finally, in exercise III we find a pattern for the entire down-left diagonal, beginning at an entry on the main diagonal of square numbers. Take 1 step down on the lower-left diagonal and subtract 1. Take 2 steps down and subtract 4 (or 2×2 or 2^2). In general, take s steps down and subtract s^2. More formally, we notice that, for each r and for each s (less than r),

$$(r+1) \times (r-1) = r^2 - 1^2.$$

Each of these patterns is graphically represented. If questions arise, resort to the sketches rather than to abstract arguments.

On practice sheet 33 pupils are asked to depend on trial and error and on reasoning. We hope that when they encounter similar examples later, in an organized algebra course, they will remember that they are not slaves to rules.



What Are My Rules? EPS-33

1.

x	5	9
x + 17	22	26

2.

m	7	9
$m^2 + 1$	50	82

3.

r	10	100
3r − 7	23	293

4.

s	8	15
$s^2 + s$	72	240

5.

a	6	4
$70 - a^2$	34	54

6.

x	9	6
$x^2 - 3x$	54	18

7.

a	5	10
a + b	8	17
a − b	2	3
b	3	7

8.

xy	21	20
2x − y	11	6
x	7	5
y	3	4

9.

mn	18	48
m	9	8
m − n	7	2
n	2	6

10.

s − r	4	2
2rs	24	70
s	6	7
r	2	5

11.

a	5	2
$a^2 + b$	28	11
b	3	7
$a + b^2$	14	51

12.

2(x − y)	22	2
3x + 3y	39	45
x	12	8
y	1	7

13.

8x	40
x	5

14.

$a^2 - 12$	24
a	6

15.

3(n + 1)	30
n	9

16.

b	8
$b^2 - 2b$	48

17.

a + b	17
a − b	13
a	15
b	2

18.

y − x	3
xy	28
x	4
y	7

19.

2f + r	11
f + 3r	18
f	3
r	5

20.

$b^2 - 2k$	60
2bk	32
b	8
k	2

A. I'm thinking of a pair of numbers. Their sum is 25 and their difference is 9. The numbers must be 17 and 8.

B. The product of a pair of numbers is 98. One is twice as large as the other. The numbers must be 14 and 7.

C. I'm thinking of a pair of numbers. If I add the square of one to the square of the other, the sum is 41. The numbers are 5 and 4.

D. Helen's age is twice that of Bill's. The sum of their ages is 27 years. Helen is 18 years old and Bill is 9 years old.

Mark the correct answer → Mark the furthest from correct answer

27 × 7	(1)	(2)	(3)	(4)
	199	189	34	207

(1) (2) (3) (4) — ○ ● ⊗ ○ — Difference between the answers you marked: 155

1 36 × 24 = 144 + 72 = 864

(1) 216
(2) 774
(3) 1,444
(4) 864

(1) (2) (3) (4) — ⊗ ○ ○ ● — 648

2 $37.28 − 18.94 = 18.34

(1) $ 56.22
(2) $ 18.34
(3) $ 28.34
(4) $ 18.94

(1) (2) (3) (4) — ⊗ ● ○ ○ — $37.88

3 1,867 + 170 + 568 + 77 = 2,682

(1) 2,826
(2) 2,862
(3) 2,628
(4) 2,682

(1) (2) (3) (4) — ○ ⊗ ○ ● — 180

4 13 | 221 = 17 (13 × 17 = 221; 13 + 91)

(1) 17
(2) 108
(3) 38
(4) 27

(1) (2) (3) (4) — ● ⊗ ○ ○ — 91

5 (3 × 18) − 18 = 36

(1) 12
(2) 18
(3) 108
(4) 39

(1) (2) (3) (4) — ○ ● ⊗ ○ — 90

6 496 × 8 = 3,968

(1) 3,228
(2) 4,968
(3) 3,968
(4) 3,758

(1) (2) (3) (4) — ○ ⊗ ● ○ — 1,000

Prices at the Supermarket

Grapefruits : 10¢ each, 3 for 28¢
Bananas : 14¢ a lb., 2 lbs. for 25¢
Oranges : 48¢ a dozen
Potatoes : 7¢ a lb.
10 lbs. 59¢
25 lbs. $1.25

7 Mr. Wilson bought a dozen grapefruit. He paid:

(1) $ 1.12 (3) 84¢
(2) $ 1.20 (4) $ 2.80

(1) (2) (3) (4) — ● ○ ○ ⊗ — $1.68

8 He also bought 30 pounds of potatoes. They cost him:

(1) $ 2.10 (3) $ 1.60
(2) $ 1.77 (4) $ 1.98

(1) (2) (3) (4) — ⊗ ○ ● ○ — 50¢

9 Mrs. Rand bought half a dozen oranges and 4 pounds of bananas. She paid:

(1) 88¢ (3) 50¢
(2) 74¢ (4) 24¢

(1) (2) (3) (4) — ○ ● ○ ⊗ — 50¢

10 What would be the cost of 5 grapefruit and 7 pounds of bananas?

(1) $ 1.48 (3) $ 1.55
(2) $ 1.20 (4) $ 1.37

(1) (2) (3) (4) — ○ ○ ⊗ ● — 18¢

11 What would be the cost of $2\frac{1}{2}$ dozen oranges, 20 pounds of potatoes, and half a dozen grapefruit?

(1) $ 3.17 (3) $ 2.90
(2) $ 2.94 (4) $ 2.49

(1) (2) (3) (4) — ○ ● ○ ⊗ — 45¢

12 What would be the total cost of 15 pounds of potatoes and 15 pounds of bananas?

(1) $ 2.83 (3) $ 4.50
(2) $ 3.25 (4) $ 5.02

(1) (2) (3) (4) — ● ○ ○ ⊗ — $2.19

34-35 Various forms of typical multiple-choice questions with an additional twist

Tasks: *Pupils complete the test items 1-12.*

Purpose: To evaluate pupil achievement in arithmetic.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: This pupil page is intended to fill several needs. Because standardized achievement tests are usually limited to multiple-choice questions and machine checking, some pupils are upset by the unfamiliar appearance and rules when they take such tests. This page gives the pupil a chance to become familiar with this test form through practice and discussion—before encountering it in a formal testing situation where practice and discussion are severely limited. Further, these items are typical of many of the more difficult questions in tests designed for sixth-grade pupils.

In the first part of each question, pupils select the correct response from four possible responses. But each question has two additional parts, both more interesting and more difficult than the first part, but dependent on it: (1) selecting the answer that is furthest from the correct answer, and (2) finding the difference between those two responses. These additional features save these questions from being typically dull and discouraging to the mind that is awake.

Finally, the list of supermarket prices suggests some problems that the slowest pupils can work at successfully and others that the most able may find challenging.

a. How much would 100 pounds of potatoes cost at each of the prices given?

7¢ a lb.	($7.00)
10 lbs. for 59¢	($5.90)
25 lbs. for $1.25	($5.00)

b. Mrs. Green spent $1.43 for fruit. What might she have bought?

6 grapefruit	56¢
3 lbs. of bananas	39¢
1 doz. oranges	48¢
	$1.43

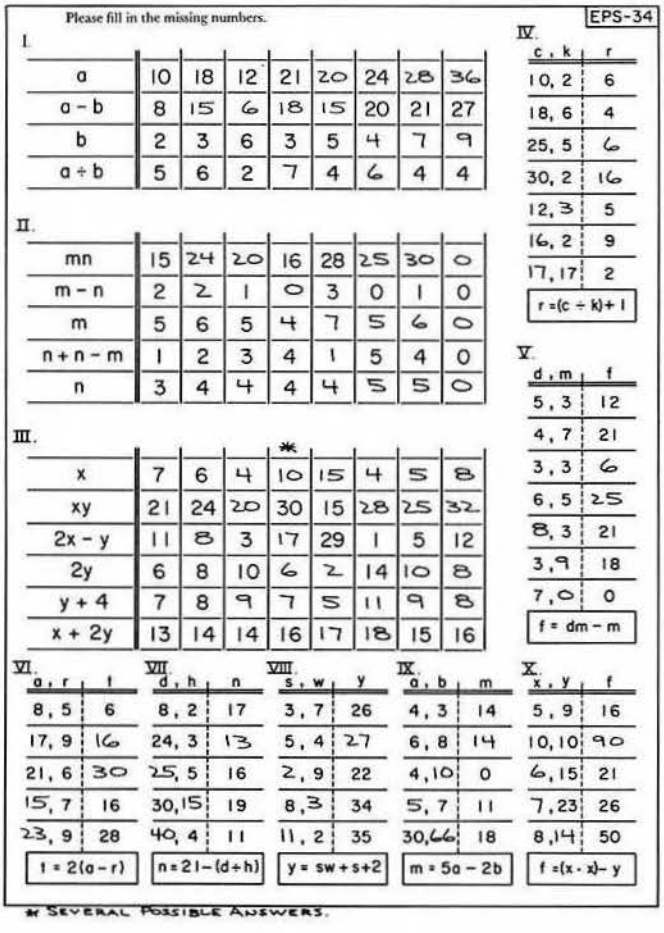
How else might she have spent $1.43 for fruit—assuming she bought only a whole number of pounds of bananas and a whole number of dozens of oranges? (2 grapefruit, 20¢; 6 lbs. of bananas, 75¢; and 1 doz. oranges, 48¢.)

Could she have spent exactly $1.43 without buying any oranges? (Yes: 12 grapefruit, $1.04, and 3 lbs. of bananas, 39¢, and that's the only way.)

c. How many different ways could a person spend exactly 75¢ for the items shown at the prices indicated (assuming, again, that only whole numbers of pounds and dozens are used)?

Pupils can be encouraged to make up and challenge the class with similar questions to which they have found the answers.

Practice sheet 34 presents a number of exercises in which items are completed by interpreting pattern indicators.



Please fill in the missing numbers.

EPS-34

I.

a	10	18	12	21	20	24	28	36
a − b	8	15	6	18	15	20	21	27
b	2	3	6	3	5	4	7	9
a ÷ b	5	6	2	7	4	6	4	4

II.

mn	15	24	20	16	28	25	30	0
m − n	2	2	1	0	3	0	1	0
m	5	6	5	4	7	5	6	0
n + n − m	1	2	3	4	1	5	4	0
n	3	4	4	4	4	5	5	0

III.

				*				
x	7	6	4	10	15	4	5	8
xy	21	24	20	30	15	28	25	32
2x − y	11	8	3	17	29	1	5	12
2y	6	8	10	6	2	14	10	8
y + 4	7	8	9	7	5	11	9	8
x + 2y	13	14	14	16	17	18	15	16

IV.

c, k	r
10, 2	6
18, 6	4
25, 5	6
30, 2	16
12, 3	5
16, 2	9
17, 17	2

r = (c ÷ k) + 1

V.

d, m	f
5, 3	12
4, 7	21
3, 3	6
6, 5	25
8, 3	21
3, 9	18
7, 0	0

f = dm − m

VI.

a, r	t
8, 5	6
17, 9	16
21, 6	30
15, 7	16
23, 9	28

t = 2(a − r)

VII.

d, h	n
8, 2	17
24, 3	13
25, 5	16
30, 15	19
40, 4	11

n = 21 − (d + h)

VIII.

s, w	y
3, 7	26
5, 4	27
2, 9	22
8, 3	34
11, 2	35

y = sw + s + 2

IX.

a, b	m
4, 3	14
6, 8	14
4, 10	0
5, 7	11
30, 66	18

m = 5a − 2b

X.

x, y	f
5, 9	16
10, 10	90
6, 15	21
7, 23	26
8, 14	50

f = (x · x) − y

* SEVERAL POSSIBLE ANSWERS.

Indicate the largest amount by writing an A in the blue block, the next largest by a B, the next by a C, etc. In the exercise compute the indicated differences.

I

three hours ..... A

one and one-quarter hours ..... B

fifty minutes C

difference in minutes between

A and B: 105 min. B and C: 25 min.

A and C: 130 min.

II

two dollars and sixty-seven cents ..... B

three-quarters of a dollar C

four and one-half dollars ..... A

difference in cents between

A and B: 183 ¢ B and C: 192 ¢

A and C: 375 ¢

III

two yards ..... A

two and one-half feet C

one yard, one foot, and seven inches ..... B

one-half yard ..... D

difference in inches between

A and B: 17 in. B and D: 37 in.

B and C: 25 in. C and D: 12 in.

A and C: 42 in. A and D: 54 in.

IV

thirty-one ounces ..... B

three-quarters of a pound ..... D

four pounds ..... A

one and one-half pounds C

difference in ounces between

A and B: 33 oz. B and D: 19 oz.

B and C: 7 oz. C and D: 12 oz.

A and C: 40 oz. A and D: 52 oz.

V

five thousand, seven hundred ..... A

five thousand, seven ..... F

five thousand, nineteen ..... E

five thousand, ninety C

five thousand, one hundred ninety ..... B

five thousand, seventy ..... D

difference between

A and F: 693 B and D: 120

E and C: 71 A and C: 610

C and D: 20 F and C: 83

A and D: 630 F and B: 183

B and E: 171 A and E: 681

VI

eight thousand, sixty-seven ..... E

forty thousand, three hundred ..... A

nine thousand, eight hundred fifty ..... D

ten thousand, nine hundred eight C

fifteen thousand ..... B

six thousand, nine ..... F

difference between

D and F: 3,841 A and B: 25,300

C and D: 1,058 C and F: 4,899

B and C: 4,092 B and D: 5,150

E and F: 2,058 C and E: 2,841

A and D: 30,450 A and C: 29,392

Tasks: *In exercises I-VI pupils first indicate the order of the measures given. Then they find the differences between indicated measures.*

Purpose: To evaluate pupil understanding of conversions in measurement and of order of large numbers.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; measurement; functions and relations.

The Lesson: The United States monetary system is relatively easy to master. Other United States systems of measurement pose great difficulties because they are not based on the decimal system; pupils need a great deal of experience with these nondecimal systems.

A great deal of this practice can be provided by exercises patterned after those on this pupil page. Write down several denominate numbers from a system of measurement or ask for examples from the class. Then have pupils arrange them in order and find the difference between pairs.

With three denominate numbers, there are three pairs to consider:

A and B, B and C, A and C

With four, there are six pairs to consider:

A and B, A and C, A and D,
B and C, B and D,
C and D

With five, there are ten pairs to consider:

A and B, A and C, A and D, A and E,
B and C, B and D, B and E,
C and D, C and E,
D and E

With six, there are fifteen pairs.

Pupil page 35 begins with four such problems, only one dealing with money. The same general method of asking questions (in V or VI) is used to check on the pupils' ability to read word notation for numbers.

Practice sheet 35 presents a second pattern that can be used to provide practice with systems of measurement. While it is not specified as a rule, all the clues given indicate that the results ought to be written using the smallest possible total number of units—thus, "78 minutes" would be rewritten as "1 hour and 18 minutes" (19 units rather than 78 units).

Similar forms can be used for subtraction and for division by a number:

weeks	days
3	0
− 1	5
1	2

weeks	days		weeks	days
4	2	÷ 3 =	1	3

EPS-35

Discover sensible rules for these games and fill in the missing numbers.

1.

Weeks	Days
3	4
1	3
5	0

2.

Days	Hr.
2	20
2	5
5	1

3.

Hr.	Min.
0	37
7	60
8	37

4.

Min.	Sec.
17	37
9	90
28	7

5.

Min.	Sec.
4	65
4	65
10	10

6.

Hr.	Min.
3	25
1	52
5	17

7.

Days	Hr.
8	24
1	36
11	12

8.

Min.	Sec.
1	81
0	17
2	38

9.

Hr.	Min.
3	121
4	61
10	2

10.

Min.	Sec.
7	47
19	120
28	47

11.

Min.	Sec.
5	60
7	240
17	0

12.

Weeks	Days	Hr.
1	6	19
1	6	19
3	6	14

13.

Hr.	Min.	Sec.
8	39	19
19	84	103
29	5	2

14.

Hr.	Min.	Sec.
2	58	100
4	59	100
8	0	20

15.

Hr.	Min.
8	35
14	15
19	27
42	17

16.

Hr.	Min.	Sec
8	42	19
1	58	46
7	37	38
18	18	43

17.

Days	Hr.	Min.
4	18	50
3	14	28
5	9	57
13	19	15

18.

Hr.	Min.	Sec
0	81	121
2	37	66
19	46	59
23	48	6

19.

Weeks	Days	Hr.	Min.	Sec.		Weeks	Days	Hr.	Min.	Sec.
1	4	3	25	50	x 2 =	3	1	6	51	40

20.

Weeks	Days	Hr.	Min.	Sec.		Weeks	Days	Hr.	Min.	Sec.
0	2	8	5	25	x 3 =	1	0	0	16	15

21.

Weeks	Days	Hr.	Min.	Sec.		Weeks	Days	Hr.	Min.	Sec.
5	5	15	30	19	x 2 =	11	4	7	0	38

V Game of Fewest Numbers (A novel approach to the usual algorithms for multiplication and long division)

Select one number from each list to find the right combination to complete the table.

I

A	0, 1, 3, 4, 6, 7, 9	6	4	0	7	9	4	3	6	1	9
B	0, 10, 21, 30, 43, 57	21	43	10	0	43	21	57	30	21	57
	Totals →	27	47	10	7	52	25	60	36	22	66

II

C	0, 1, 2, 5, 7, 8	2	7	5	0	8	1	8	7
D	0, 9, 18, 28, 37, 47	18	9	0	18	47	37	28	37
E	0, 56, 113, 169, 225, 284	56	0	169	113	225	284	56	169
	Totals	76	16	174	131	280	322	92	213

III

					*	*			
F	0, 2, 4, 6, 8	4	6	2	0	8	6	4	6
G	0, 9, 19, 27, 36	36	0	9	27	19	9	27	19
H	0, 45, 93, 138, 184	93	184	45	93	45	0	138	138
I	0, 237, 466, 695, 924	237	695	924	695	466	466	237	0
	Totals	370	885	980	815	538	481	406	163

* SEVERAL ANSWERS POSSIBLE.

IV

J	0, 1, 2, 4	4	1	0	2	1	4	2
K	0, 5, 11, 16	11	5	16	16	11	0	16
L	0, 21, 43, 67	67	21	67	0	43	21	21
M	0, 88, 179, 267	0	88	0	267	88	267	179
N	0, 358, 705, 1059	358	705	358	1,059	705	358	1,059
	Totals	440	820	441	1,344	848	650	1,277

Tom announced: "I've got a way to find these combinations quickly. I simply take the largest one I can from the bottom list; next, the largest one from the next list, etc. I keep a running account of how many I have left to go:

Total	1,277
1059	Largest from List N
218	. . . amount needed
179	Largest from List M
39	. . . amount still needed
21	Largest from List L
18	. . . amount still needed
16	Largest from List K
2	. . . amount still needed
2	From List J

Use lists J, K, L, M, and N to try Tom's system on the following:

	V	VI	VII
Totals	957	672	1,206
N	705	358	1,059
	252	314	147
M	179	267	88
	73	47	59
L	67	43	43
	6	4	16
K	5	0	16
	1	4	0
J	1	4	0

Tasks: *Pupils complete the tables by selecting numbers from the indicated lists so that the sum of those selected is a specified number.*

Purpose: To provide experiences in which pupils explore a strategy like that used in the usual long division algorithm.

Unifying Ideas: Structure; sets; addition and subtraction; multiplication and division; functions and relations.

The Lesson: In each example, one must select a number from each list so that the sum of those selected is a specified number. These lists are designed to force a similar strategy in each case. In each list, all the entries except zero are larger than those in the preceding list. The pupil quickly discovers that he must first find the largest number he can that is less than the specified sum; then, when he has found out how much remains, he must look for the next largest number that is less than this remainder. He continues this process until his remainder is zero.

This is precisely the strategy upon which the usual long division algorithm is based.
Consider this example:

$$18\overline{)666}$$

Our list is this list of multiples of 10: 0, 10, 20, 30, 40, 50, 60, 70, 80, and 90. We must find a "trial quotient"; we select the largest number in the list that, when multiplied by 18, will be less than 666. We find that 40 is too large, but that 30 is less. So we proceed:

$$\begin{array}{r} 30 \\ 18\overline{)666} \\ \underline{540} \\ 126 \end{array}$$

and find that this step has accounted for 540 of the 666, leaving 126 yet to be accounted for.

Now we search in the next list—0 through 9—for the largest number that, when multiplied by 18, will be less than 126. We find that 8 would be too large, but that 7 is just right.

$$\begin{array}{r} 37 \\ 18\overline{)666} \\ \underline{540} \\ 126 \\ \underline{126} \\ 0 \end{array}$$

It is this basic strategy we wish to use in quite unfamiliar contexts before exploring how it is used in the usual division algorithm. We might call it the "biggest chunk first" strategy.

Practice sheet 36 considers this plan in a multiplication context.

EPS-36

A clerk in a department store wanted to save time computing the purchase prices of several items of the same price. Here is one of the cards he worked out for items selling at 17¢ each. Please complete it.

17¢ each			
1	$ .17	10	$ 1.70
2	.34	20	3.40
3	.51	30	5.10
4	.68	40	6.80
5	.85	50	8.50
6	1.02	60	10.20
7	1.19	70	11.90
8	1.36	80	13.60
9	1.53	90	15.30

If he wanted to know the price for 45 items at 17¢ each, he would select the appropriate entries in the chart and add them.

A. 40 at 17¢--$ 6.80
5 at 17¢--$ .85
45 at 17¢--$ 7.65

Here are two more examples:

B. 30 at 17¢--$ 5.10
6 at 17¢--$ 1.02
36 at 17¢--$ 6.12

C. 20 at 17¢--$ 3.40
7 at 17¢--$ 1.19
27 at 17¢--$ 4.59

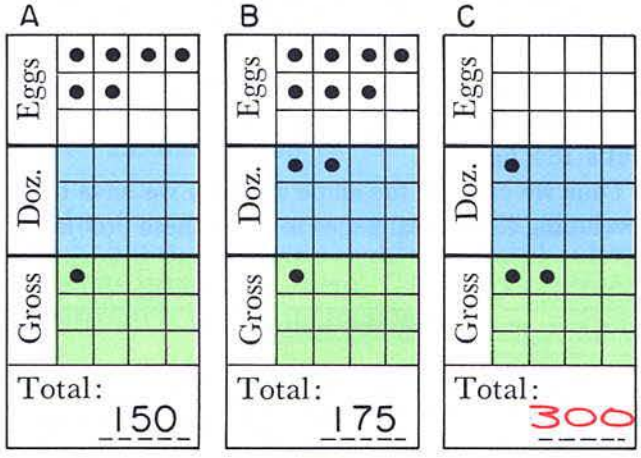
Use the chart to help you compute the following:

1. 13 at 17¢--$ 2.21
2. 8 at 17¢--$ 1.36
3. 70 at 17¢--$ 11.90
4. 18 at 17¢--$ 3.06
5. 64 at 17¢--$ 10.88
6. 19 at 17¢--$ 3.23
7. 91 at 17¢--$ 15.47
8. 86 at 17¢--$ 14.62
9. 62 at 17¢--$ 10.54
10. 73 at 17¢--$ 12.41
11. 35 at 17¢--$ 5.95
12. 44 at 17¢--$ 7.48
13. 77 at 17¢--$ 13.09
14. 99 at 17¢--$ 16.83
15. 100 at 17¢--$ 17.00
16. 40 at 17¢--$6.80
17. 21 at 17¢--$3.57
18. 9 at 17¢--$1.53
19. 15 at 17¢--$2.55
20. 22 at 17¢--$3.74
21. 17 at 17¢--$2.89
22. 39 at 17¢--$6.63
23. 56 at 17¢--$9.52
24. 78 at 17¢--$13.26
25. 24 at 17¢--$ 4.08
26. 47 at 17¢--$ 7.99
27. 94 at 17¢--$15.98
28. 7 at 34¢--$ 2.38
29. 40 at 34¢--$13.60
30. 47 at 34¢--$15.98
31. 60 at 34¢--$ 20.40
32. 70 at 34¢--$ 23.80
33. 130 at 34¢--$ 44.20
34. 21 at 51¢--$ 10.71
35. 66 at 68¢--$ 44.88
36. 90 at 68¢--$ 61.20

Mr. Stone operated a chicken ranch. His main business was supplying eggs to the stores in his vicinity.

He packed eggs by the dozen and by the gross. (A gross is 12 dozen.) If a customer ordered 15 dozen eggs, he would ship 1 gross and 3 dozen.

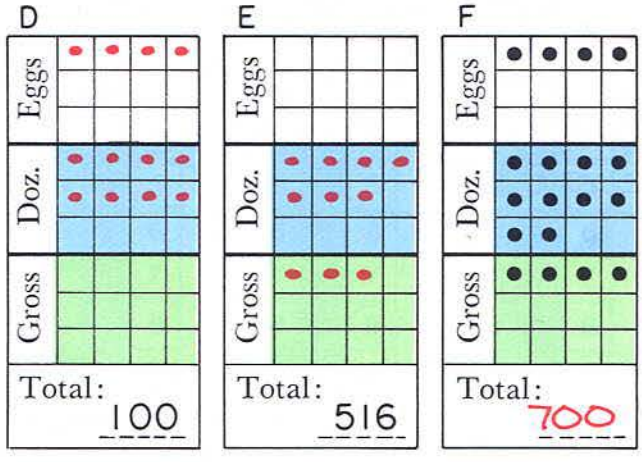
Here are examples of special order cards that were filled when orders came in:



Please fill in the total number of eggs on order card C above.

The shipping department would use these cards to determine how many gross, how many dozen, and how many additional eggs (an incomplete dozen) to send to the customer whose name was on the back of the card.

Please show how cards D, E, and F should be marked.

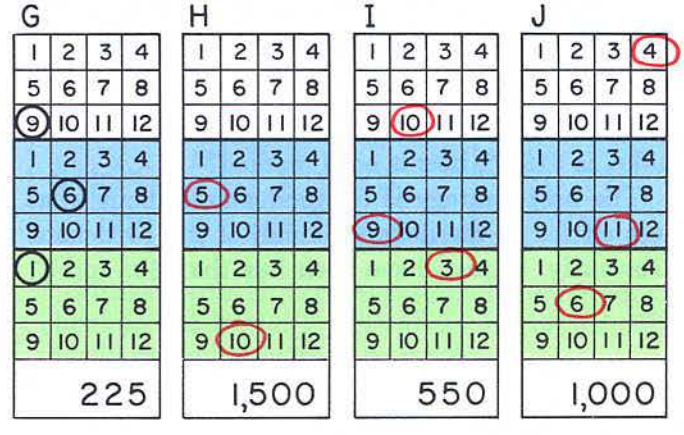


As a time saver, someone in the office made a chart similar to the following. Please supply the missing entries.

Incomplete Dozen	Dozen	Gross
1	12	144
2	24	288
3	36	432
4	48	576
5	60	720
6	72	864
7	84	1,008
8	96	1,152
9	108	1,296
10	120	1,440
11	132	1,584

An assistant to Mr. Stone designed a card that was easier to use. He told Mr. Stone, "Everyone knows where we mark the incomplete dozens, the dozens, the gross, and total. We can leave the words out. Also, we can circle a number faster than we can put in dots."

Please complete the new order cards G, H, I, and J. Card G is already completed to illustrate the simplified method. (The chart above may help.)



G: 1 gross, 6 dozen, and 9 eggs; or 225 eggs

37 Exploring the games using "gross, dozens, and eggs" punch cards

Tasks: *Pupils complete the records of orders of eggs.*

Purpose: To provide experiences in which pupils explore a strategy like that used in the usual long division algorithm.

Unifying Ideas: Structure; sets; number and counting; numeration; addition and subtraction; multiplication and division; measurement.

The Lesson: The three lists considered on pupil page 37 have the same pattern as those on page 36; each contains numbers larger than the previous list, but now there is a definite pattern to each list. The lists are:

1, 2, 3, 4, ... 11
12, 24, 36, 48, ... 132
144, 288, ... 1,584

Each number in the second and third lists is 12 times the corresponding number in the preceding list.

You may wish to consider the lists suggested by the calendar (if all months had 30 days):

D: 0, 1, 2, 3, 4, 5, 6
W: 0, 7, 14, 21, 28
M: 0, 30, 60, 90, 120, ...

Using only the numbers in the lists, partition the given total number of days:

	M	W	D
18	0	14	4
29	0	28	1
42	30	7	5
etc.			

Thus, the first row tells us that 18 days consists of 14 days and 4 days, or 2 weeks and 4 days.

These lists are suggested by a yardstick:

I: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11
F: 0, 12, 24
Y: 0, 36, 72, 108, 144, 180, 216, ...

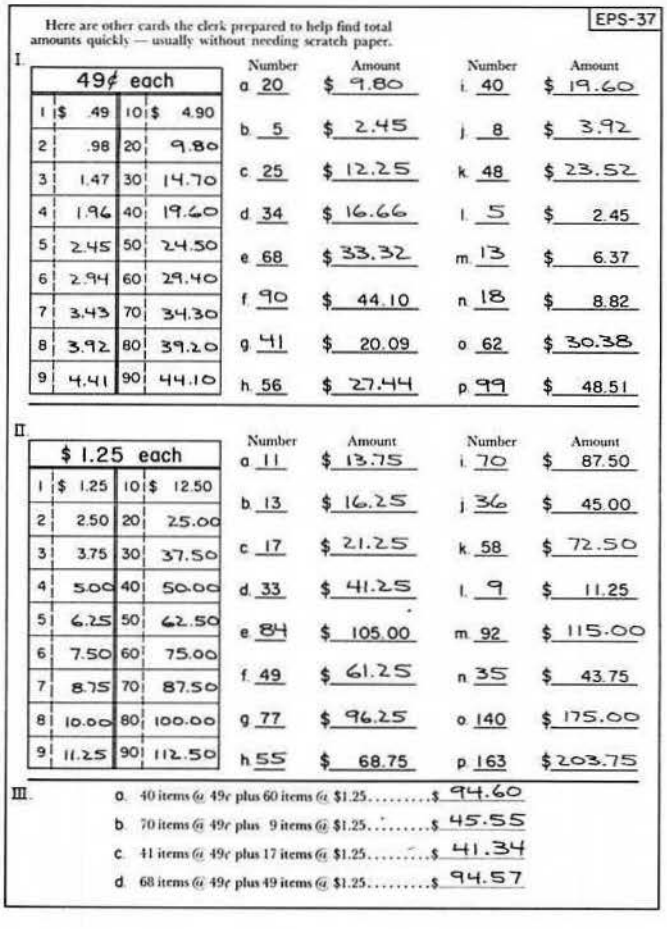
Using only the numbers in the list, partition the given total number of inches:

	Y	F	I
27	0	24	3
19	0	12	7
40	36	0	4
130	108	12	10
150	144	0	6
204	180	24	0
etc.			

Thus, the sixth row tells us that 204 inches consists of 180 inches and 24 inches, or 5 yards and 2 feet.

Any system of measurement can be studied in this way. Practice sheets 36 through 39 investigate lists derived from our money system.

In selecting the appropriate items in all these activities, pupils will be led to use the same basic strategy—to select successive steps that progressively cut the problem down in size. This is the basic strategy we use in the usual long division algorithm.



Here are other cards the clerk prepared to help find total amounts quickly — usually without needing scratch paper.

EPS-37

I.

49¢ each			
1	$ .49	10	$ 4.90
2	.98	20	9.80
3	1.47	30	14.70
4	1.96	40	19.60
5	2.45	50	24.50
6	2.94	60	29.40
7	3.43	70	34.30
8	3.92	80	39.20
9	4.41	90	44.10

	Number	Amount	Number	Amount
a	20	$ 9.80	i. 40	$ 19.60
b	5	$ 2.45	j. 8	$ 3.92
c	25	$ 12.25	k. 48	$ 23.52
d	34	$ 16.66	l. 5	$ 2.45
e	68	$ 33.32	m. 13	$ 6.37
f	90	$ 44.10	n. 18	$ 8.82
g	41	$ 20.09	o. 62	$ 30.38
h	56	$ 27.44	p. 99	$ 48.51

II.

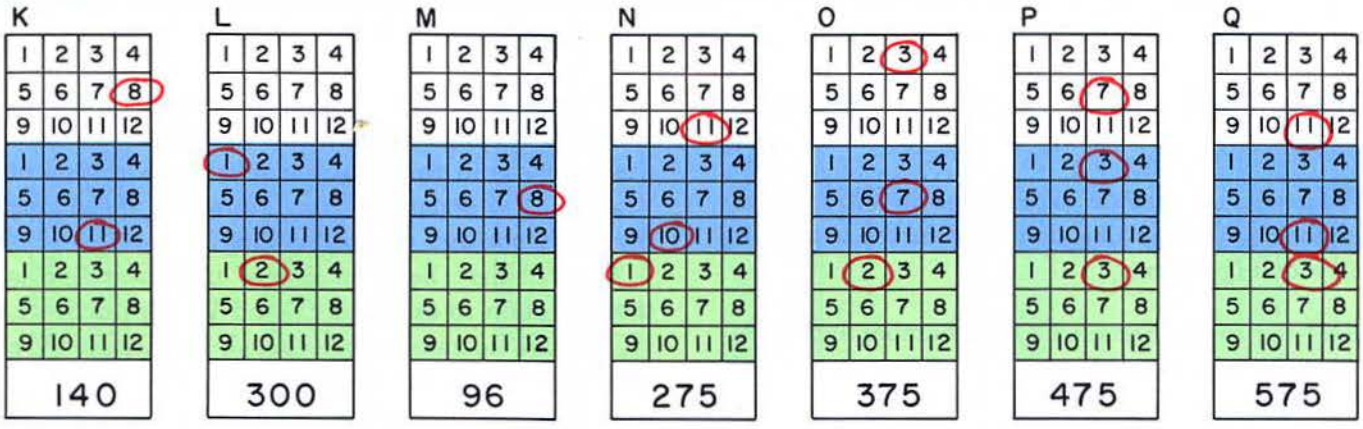
$ 1.25 each			
1	$ 1.25	10	$ 12.50
2	2.50	20	25.00
3	3.75	30	37.50
4	5.00	40	50.00
5	6.25	50	62.50
6	7.50	60	75.00
7	8.75	70	87.50
8	10.00	80	100.00
9	11.25	90	112.50

	Number	Amount	Number	Amount
a	11	$ 13.75	i. 70	$ 87.50
b	13	$ 16.25	j. 36	$ 45.00
c	17	$ 21.25	k. 58	$ 72.50
d	33	$ 41.25	l. 9	$ 11.25
e	84	$ 105.00	m. 92	$ 115.00
f	49	$ 61.25	n. 35	$ 43.75
g	77	$ 96.25	o. 140	$ 175.00
h	55	$ 68.75	p. 163	$ 203.75

III.

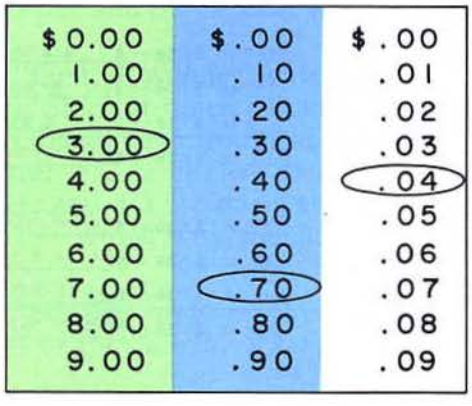
a. 40 items @ 49¢ plus 60 items @ $1.25 ..... $ 94.60
b. 70 items @ 49¢ plus 9 items @ $1.25 ..... $ 45.55
c. 41 items @ 49¢ plus 17 items @ $1.25 ..... $ 41.34
d. 68 items @ 49¢ plus 49 items @ $1.25 ..... $ 94.57

Please complete the following orders for eggs:



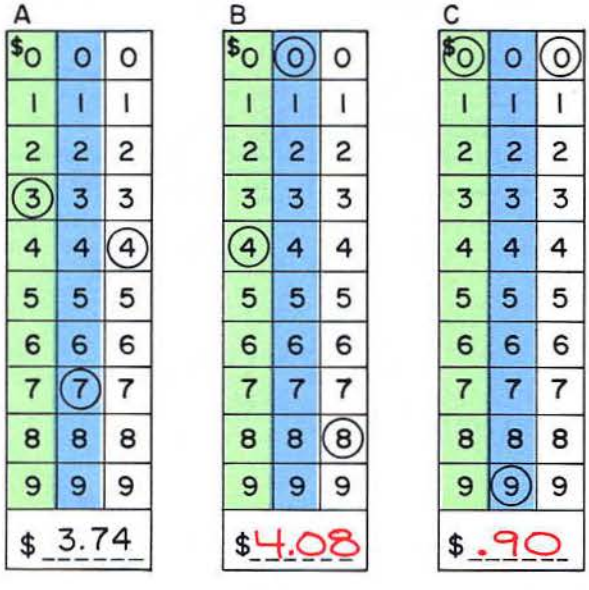
One of Mr. Stone's customers was Mrs. Carter's Lunch Room.

Mrs. Carter liked the new egg order cards so much that she decided to make checks that had amounts printed on them. The clerks learned how to circle the right amounts:



This is marked to show $3.74

Mrs. Carter simplified the forms:

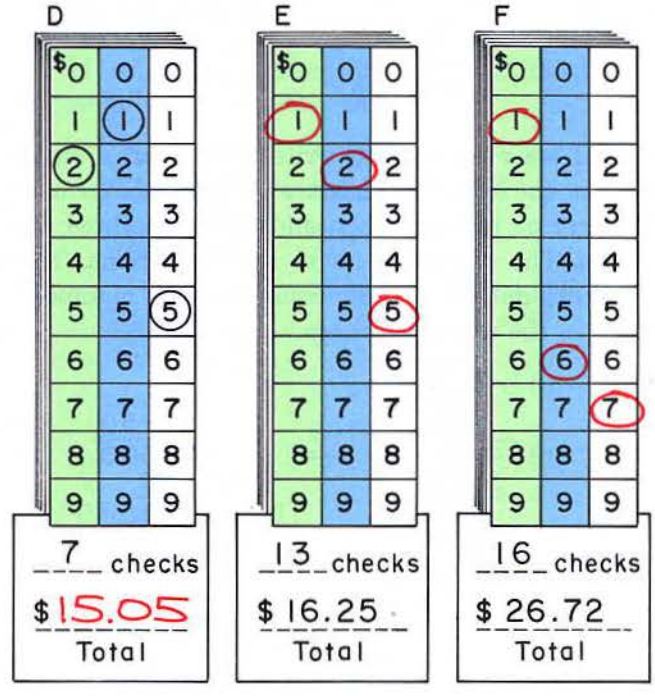


At the end of the day, she often collected the checks that were for the same amount of money and put them in a pile. For each pile, she made a note of

(1) the number of checks that were in the pile, and

(2) the total for the pile.

Please fill in the missing information below.



Please circle the correct amounts on the top check in the E and F piles above in the way all checks in each pile must be marked.

Tom was excited: "I can use my system again."

(continued)

38-39 Exploring the games using "dollars, dimes, and cents" punch cards

Tasks: *Pupils complete the records of amounts of* money.

Purpose: To provide experiences in which pupils explore a strategy like that upon which the usual long division algorithm is based.

Unifying Ideas: Structure; sets; number and counting; numeration; addition and subtraction; multiplication and division; measurement.

The Lesson: You may wish to discuss briefly examples N-Q. Since the totals increase by 100, one might reasonably expect to find a simple pattern he could follow, but this is not the case. In terms of the dozens-gross system, 100 eggs is 8 dozen and 4 eggs. Had the total numbers of eggs increased by 144 (say 78, 222, 366, and 510), the exercises would have been very easy.

This difficulty disappears in a system based on multiples of 10. We find it very easy to express the total number of pennies indicated on Mrs. Carter's checks.

"Suppose Mrs. Carter's clerks tried to indicate the same amount in two different ways. Could they do it without circling more than one number in each column?" Many pupils will ponder this question for some time and perhaps try to find a way. Each attempt will bring them squarely in contact with a fundamental pattern of the decimal system of numeration.

Exercise D asks: "What is the total of 7 checks if each is $2.15?" If the pupils think about this as

$2.00 circled on 7 checks = $14.00
.10 circled on 7 checks = .70
.05 circled on 7 checks = .35
$2.15 circled on 7 checks = $15.05

that will help them think in reverse when they are asked in E, "If the total of 13 checks, all marked alike, is $16.25, how is each marked?"

The best strategy is to wonder first about what was circled in the dollars column of the 13 tickets. Obviously, it must have been less than $2; and $1 is not too much. So, $1 circled on 13 tickets takes care of $13 out of $16.25.

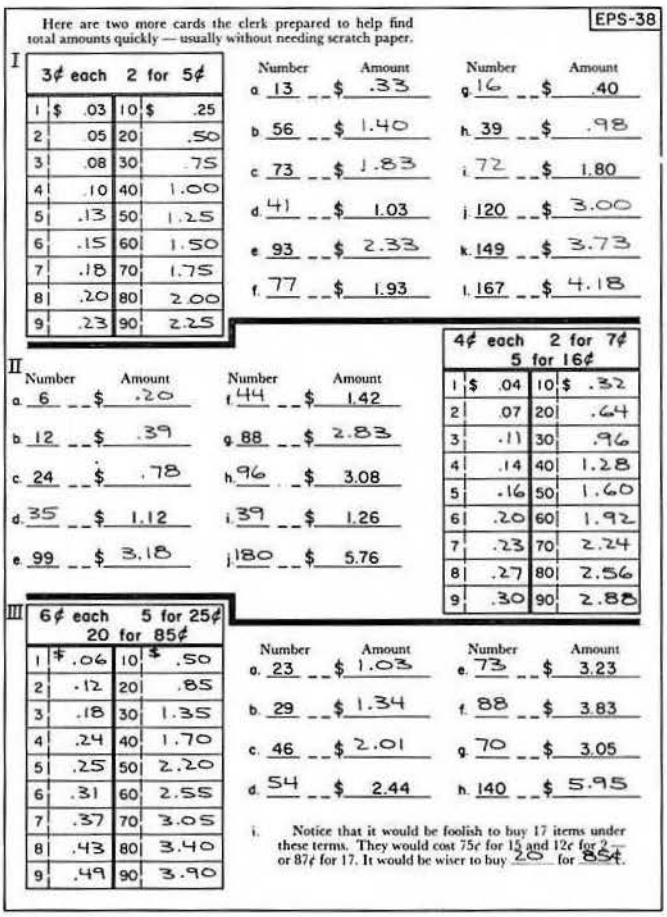
$16.25
— 13.00
$ 3.25 to be accounted for

Now, if we consider the dimes column, we notice that

13 × 10¢ = $1.30
13 × 20¢ = 2.60—not too much
13 × 30¢ = 3.90—too much

So 20¢ must have been circled on all 13 tickets, accounting for all but $3.25—$2.60, or 65¢. And that's 13 × 5¢.

Thus we can use the same strategy we have been developing for several pages to solve these problems.



Here are two more cards the clerk prepared to help find total amounts quickly — usually without needing scratch paper.

EPS-38

I

3¢ each		2 for 5¢	
1	$ .03	10	$ .25
2	.05	20	.50
3	.08	30	.75
4	.10	40	1.00
5	.13	50	1.25
6	.15	60	1.50
7	.18	70	1.75
8	.20	80	2.00
9	.23	90	2.25

Number	Amount	Number	Amount
a. 13	$ .33	g. 16	$ .40
b. 56	$ 1.40	h. 39	$ .98
c. 73	$ 1.83	i. 72	$ 1.80
d. 41	$ 1.03	j. 120	$ 3.00
e. 93	$ 2.33	k. 149	$ 3.73
f. 77	$ 1.93	l. 167	$ 4.18

II

Number	Amount	Number	Amount
a. 6	$ .20	f. 44	$ 1.42
b. 12	$ .39	g. 88	$ 2.83
c. 24	$ .78	h. 96	$ 3.08
d. 35	$ 1.12	i. 39	$ 1.26
e. 99	$ 3.18	j. 180	$ 5.76

4¢ each		2 for 7¢ 5 for 16¢	
1	$ .04	10	$ .32
2	.07	20	.64
3	.11	30	.96
4	.14	40	1.28
5	.16	50	1.60
6	.20	60	1.92
7	.23	70	2.24
8	.27	80	2.56
9	.30	90	2.88

III

6¢ each		5 for 25¢ 20 for 85¢	
1	$ .06	10	$ .50
2	.12	20	.85
3	.18	30	1.35
4	.24	40	1.70
5	.25	50	2.20
6	.31	60	2.55
7	.37	70	3.05
8	.43	80	3.40
9	.49	90	3.90

Number	Amount	Number	Amount
a. 23	$ 1.03	e. 73	$ 3.23
b. 29	$ 1.34	f. 88	$ 3.83
c. 46	$ 2.01	g. 70	$ 3.05
d. 54	$ 2.44	h. 140	$ 5.95

i. Notice that it would be foolish to buy 17 items under these terms. They would cost 75¢ for 15 and 12¢ for 2— or 87¢ for 17. It would be wiser to buy 20 for 85¢.

"Examine the piles of checks I-VI. Then I'll tell you how I find the missing information."



"I need to account for $39.95, the total of the checks in pile I. There are 17 checks in this pile and they are all alike.

"First, I wonder which number should be circled in the green column of each check. If 1 is circled, that would account for $17 of the total; if 2 is circled, that would account for $34; and if 3 is circled, that would account for $51 — which is too much.

"So, 2 must be circled. This accounts for $34 of the $39.95. I keep notes like this:

$ 39.95 ..... Total
34.00 ..... from green column
5.95 ..... to be accounted for

"Next, I consider the blue (dimes) column:

$ .10 x 17 = $ 1.70
.20 x 17 = 3.40
.30 x 17 = 5.10
.40 x 17 = 6.80 (too much!)

"Now I know that 3 must be circled in the blue column. I note the following:

$ 5.95 ..... (to be accounted for)
5.10 ..... from blue column
.85 ..... still to be accounted for

"So, 5 must be circled in the white column, since $.05 × 17 = $.85.

"Here are my notes on piles II and III." Please fill in the missing entries.

$ 25.76 ..... total of 14 checks
14.00 ..... $1.00 × 14
11.76 ..... to be accounted for
11.20 ..... $0.80 × 14
.56 ..... to be accounted for
.56 ..... $0.04 × 14

"Pile number III:

$ 40.92 ..... total of 11 checks
33.00 ..... $3.00 × 11
7.92
7.70 ..... $0.70 × 11
.22
.22 ..... $0.02 × 11

Please find the missing entries for piles IV through VI.

Tasks: *Pupils complete records of amounts of money.*

Purpose: To provide experiences in which pupils explore a strategy like that used in the usual long division algorithm.

Unifying Ideas: Structure; sets; number and counting; numeration; addition and subtraction; multiplication and division; measurement.

The Lesson: For pupils still hesitant about problems like those on pupil page 39, consider putting on the board an outline of the steps being taken. As you already realize, this form comes very close to the algorithm we wish all pupils to learn.

18 checks | $45.00 ... Total

36.00 ... $2.00 × 18

9.00

9.00 ... $0.50 × 18

00

___ ... $0.00 × 18

Each check: $2.50

Would we ever have to write more than a single digit in the little square boxes? Would we ever have to carry or regroup when adding in the three columns? The answer is no to both questions. This helps explain why we can condense such manipulations in the usual algorithm.

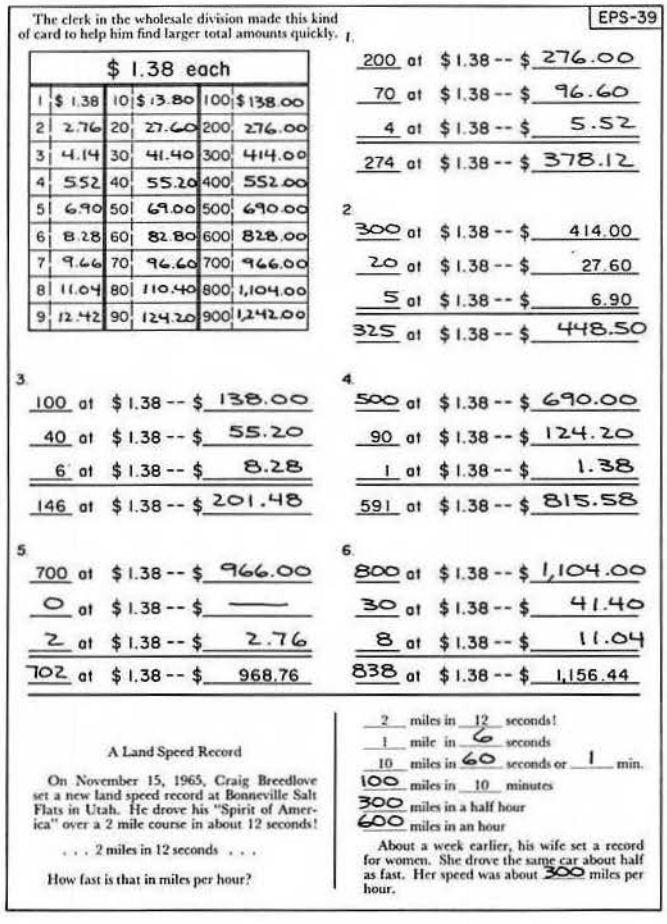
```
      0
     50
    200
18 ) 4500
     3600
      900
      900
        0
        0
```

→

```
      250
18 ) 4500
     36
      90
      90
       0
```

It would be useless for pupils to master the form presented on pupil page 39. But their ability to work the problems in this way is a reliable indication of whether they grasp the overall strategy of the long division algorithm we are building.

A particular check on pupils' understanding is their reaction to exercises IV and V. In exercises I, II, and III no zero is ever circled. In exercise IV a zero must be circled in the pennies column. In V a zero must be circled in the dollars column. Why? Clearly because the total of 31 checks is less than $31.00, so the amount of each must be less than $1.00. In fact, it is 95¢, or $0.95.



The clerk in the wholesale division made this kind of card to help him find larger total amounts quickly.

EPS-39

$1.38 each

1	$1.38	10	$13.80	100	$138.00
2	2.76	20	27.60	200	276.00
3	4.14	30	41.40	300	414.00
4	5.52	40	55.20	400	552.00
5	6.90	50	69.00	500	690.00
6	8.28	60	82.80	600	828.00
7	9.66	70	96.60	700	966.00
8	11.04	80	110.40	800	1,104.00
9	12.42	90	124.20	900	1,242.00

1.

200 at $1.38 -- $ 276.00

70 at $1.38 -- $ 96.60

4 at $1.38 -- $ 5.52

274 at $1.38 -- $ 378.12

2.

300 at $1.38 -- $ 414.00

20 at $1.38 -- $ 27.60

5 at $1.38 -- $ 6.90

325 at $1.38 -- $ 448.50

3.

100 at $1.38 -- $ 138.00

40 at $1.38 -- $ 55.20

6 at $1.38 -- $ 8.28

146 at $1.38 -- $ 201.48

4.

500 at $1.38 -- $ 690.00

90 at $1.38 -- $ 124.20

1 at $1.38 -- $ 1.38

591 at $1.38 -- $ 815.58

5.

700 at $1.38 -- $ 966.00

0 at $1.38 -- $ ___

2 at $1.38 -- $ 2.76

702 at $1.38 -- $ 968.76

6.

800 at $1.38 -- $ 1,104.00

30 at $1.38 -- $ 41.40

8 at $1.38 -- $ 11.04

838 at $1.38 -- $ 1,156.44

A Land Speed Record

On November 15, 1965, Craig Breedlove set a new land speed record at Bonneville Salt Flats in Utah. He drove his "Spirit of America" over a 2 mile course in about 12 seconds!

. . . 2 miles in 12 seconds . . .

How fast is that in miles per hour?

2 miles in 12 seconds!

1 mile in 6 seconds

10 miles in 60 seconds or 1 min.

100 miles in 10 minutes

300 miles in a half hour

600 miles in an hour

About a week earlier, his wife set a record for women. She drove the same car about half as fast. Her speed was about 300 miles per hour.

"I've got some lists that I can use any time I run into certain problems. If I need to divide by 23, I take out my 'twenty-three' lists. Here they are."

		0	100	200	300	400	500	600	700	800	900
A	23	0	2,300	4,600	6,900	9,200	11,500	13,800	16,100	18,400	20,700

		0	10	20	30	40	50	60	70	80	90
B	23	0	230	460	690	920	1,150	1,380	1,610	1,840	2,070

		0	1	2	3	4	5	6	7	8	9
C	23	0	23	46	69	92	115	138	161	184	207

"Here is how I do the following problems."

(1) 247) 23 | 5,681

(2) 476) 23 | 10,948

(3) 759) 23 | 17,457

(4) 808) 23 | 18,584

"I select three numbers — one from List A, one from List B, one from List C — so that their sum is the dividend. I notice the number above each selection and write it in the right-hand column."

(1)

A →	4,600	200
B →	920	40
C →	161	7
	5,681	247

(2)

9,200	400
1,610	70
138	6
10,948	476

(3)

16,100	700
1,150	50
207	9
17,457	759

(4)

18,400	800
0	0
184	8
18,584	808

Please use Martin's system to complete each of the 4 examples below:

(5) 163) 23 | 3,749

(6) 397) 23 | 9,131

(7) 580) 23 | 13,340

(8) 99) 23 | 2,277

(5)

A →	2,300	100
B →	1,380	60
C →	69	3
	3,749	163

(6)

6,900	300
2,070	90
161	7
9,131	397

(7)

11,500	500
1,840	80
0	0
13,340	580

(8)

0	0
2,070	90
207	9
2,277	99

Tasks: *Pupils complete the lists of multiples of 23 and then use the results to complete the eight long division problems.*

Purpose: To provide experiences that point up the multiplication aspect of the long division algorithm.

Unifying Ideas: Structure; number and counting; numeration; addition and subtraction; multiplication and division.

The Lesson: Pupil pages 40 through 49 move steadily toward the usual long division algorithm with all its variations. This algorithm, of course, involves extensive use of multiplication. Since some pupils may still not be fluent with multiplication—hesitating often and prone to make errors—we have designed activities that point up the multiplication aspect. We begin by making lists that include the results of multiplications that will arise in the course of the division work. Further, we follow the basic strategy of finding the most effective way to cut the problem down in size.

All examples on pupil pages 40 and 41 deal with division in which 23 is the divisor. Lists A, B, and C include all the multiplications that will be involved. After these lists are completed, the examples follow a single pattern:

a. Find the largest number in any list that is not larger than the dividend. This will, of course, be a multiple of 23.

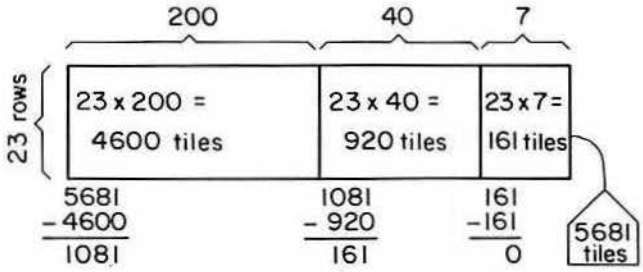
b. Subtract this number from the dividend to find the amount still to be accounted for.

c. Repeat *a* and *b* until the entire dividend has been accounted for.

Perhaps a sketch will clarify the procedure. Consider the problem

$$23\overline{)5681}$$

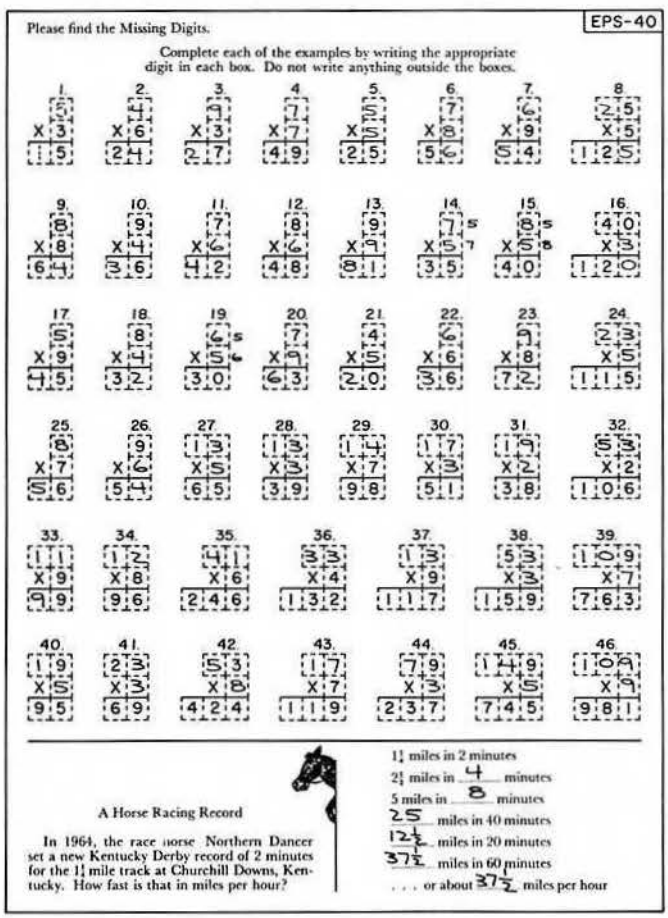
which might be asking for the number of tiles in each of 23 rows of a rectangular array consisting of 5,681 tiles.



So there must be 247 tiles in each row. But let us check. Ten rows would contain 2,470 tiles and 20 rows would contain twice as many, or 4,940. Three rows with 247 each would contain 247×3, or 741 tiles.

$$\begin{array}{r} 4940 \text{ tiles} \\ 741 \text{ tiles} \\ \hline 5681 \text{ tiles} \end{array}$$

Practice sheet 40 provides a puzzle-like atmosphere in which to consider one-digit multipliers—as used in long division.

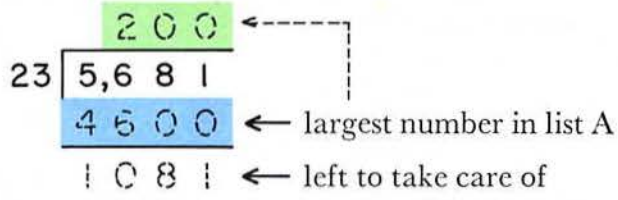


Martin Develops a Shorthand

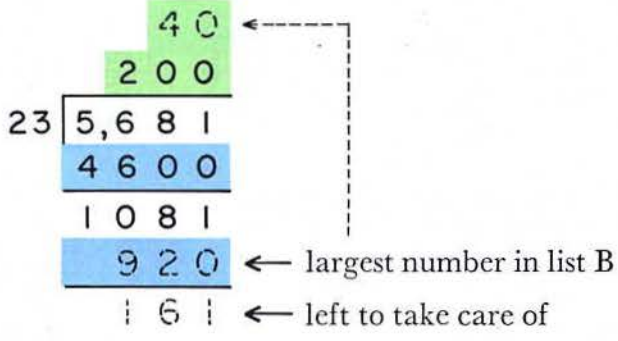
"I found a shorter way to use my system. I keep track as I go along.

"Consider the first example in the previous group. I look for the largest number in any list that is less than the dividend. I write it down this way — and find out how much I have left to take care of.

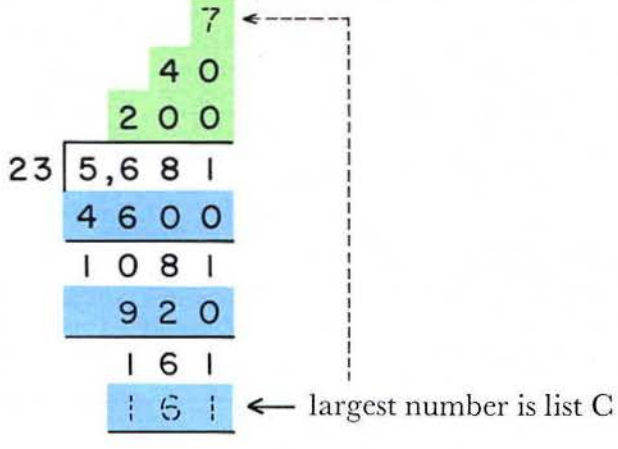
(1)



"Next, I find the largest number I can use in list B, and extend my record.

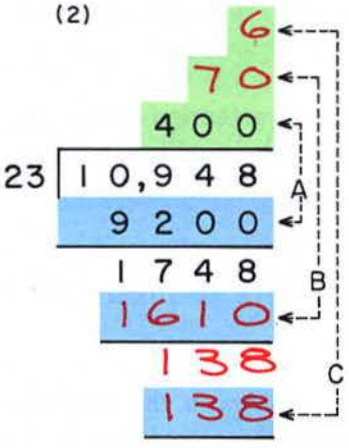


"Then

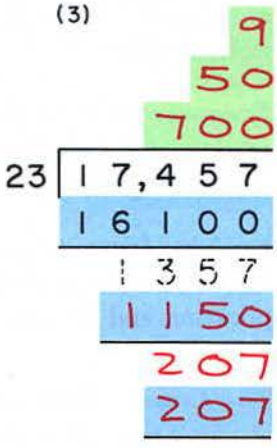


"Here is the way I would find the quotient in some of the other examples."

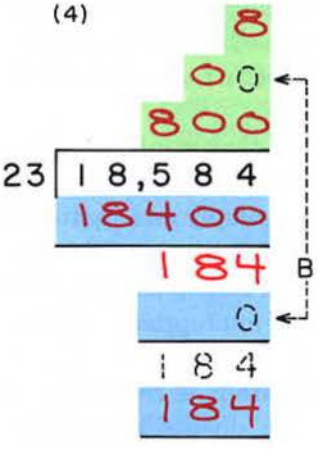
(2)



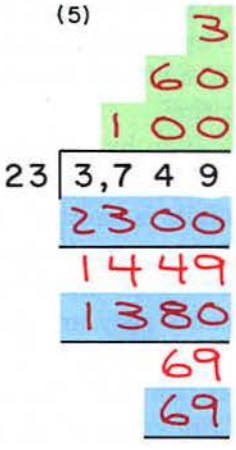
(3)



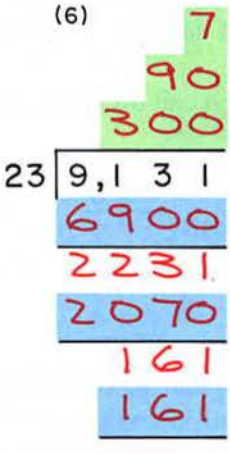
(4)



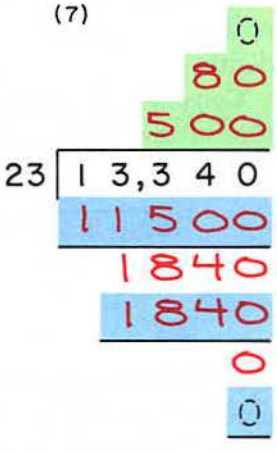
(5)



(6)



(7)



Alice objected: "But Martin, your system only works when you can use your 23-lists!"

"No, Alice, I have other lists — an 18-list, for example."

(continued on next page)

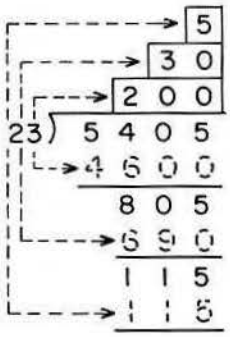
41-45 Martin develops a shorthand

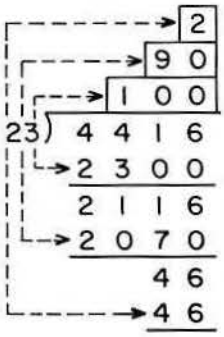
Tasks: *Pupils complete the long division problems using a method to keep track of their progress.*

Purpose: To provide practice in keeping a record of the steps in long division.

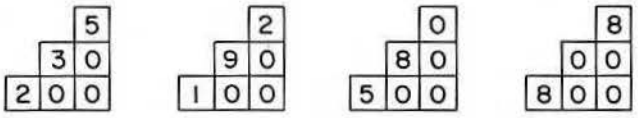
Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division.

The Lesson: Pupil page 41 considers the same problem as pupil page 40, but introduces another method of keeping track of progress. This reorganization of the record leaves us only a few short steps from the full algorithm, with all the multiplication done in advance. As the pupils show confidence with these examples, try working some in reverse.

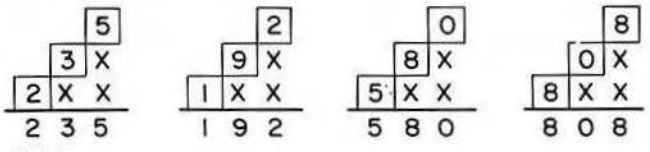




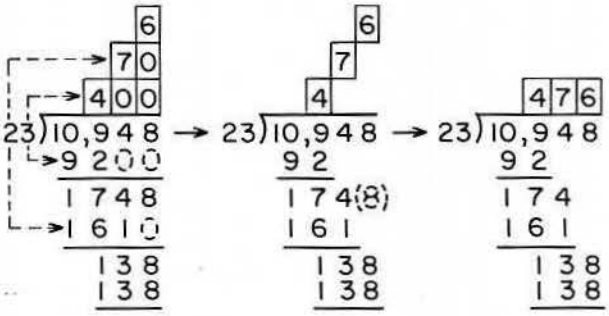
Anticipating developments ahead, let's find those positions that in every example are occupied by a 0. First, consider several pyramids of multipliers:



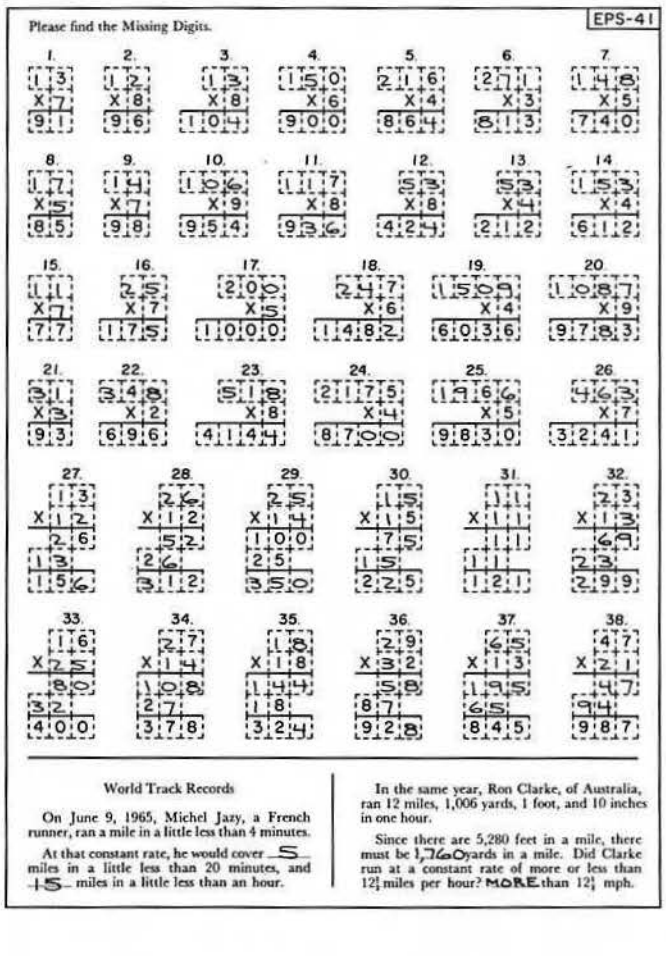
Because of the way lists A, B, and C are prepared, we know that only the left-most digit in each row can be a nonzero digit.



The zero digits produce other zero digits in the multiplication:



Practice sheet 41 continues to provide examples of multiplication that require thought as well as memory. There are enough clues given in each case so that only one reconstruction is possible.



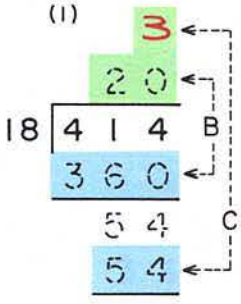
"Here are my 18-lists," Martin explained.

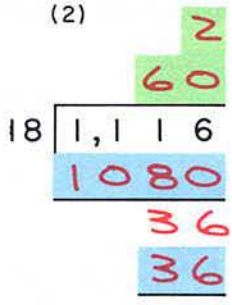
		0	100	200	300	400	500	600	700	800	900
A	18	0	1,800	3,600	5,400	7,200	9,000	10,800	12,600	14,400	16,200

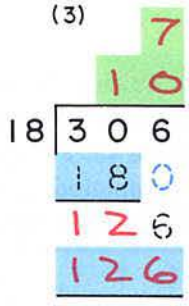
		0	10	20	30	40	50	60	70	80	90
B	18	0	180	360	540	720	900	1,080	1,260	1,440	1,620

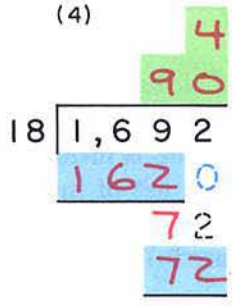
		0	1	2	3	4	5	6	7	8	9
C	18	0	18	36	54	72	90	108	126	144	162

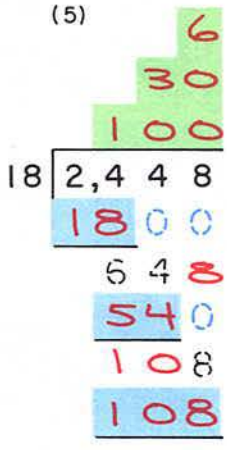
"Notice that I begin to leave out zeros in those places where I know zeros would normally be written!"

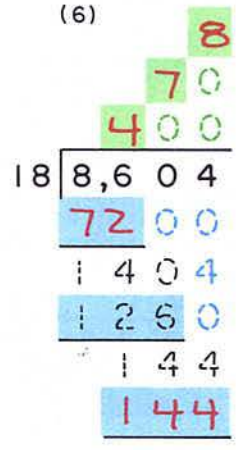
(1)


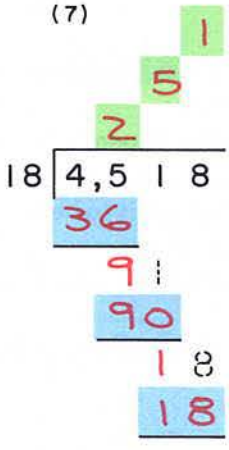
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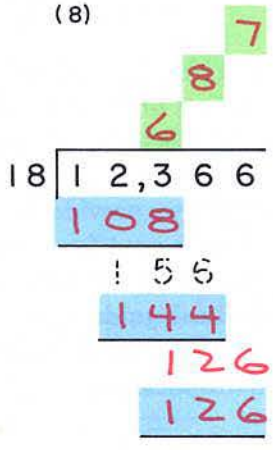
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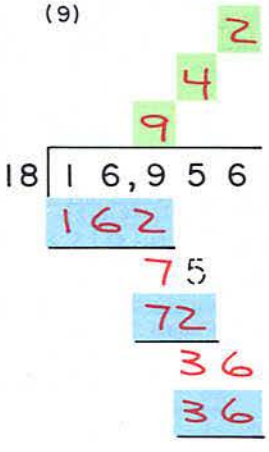
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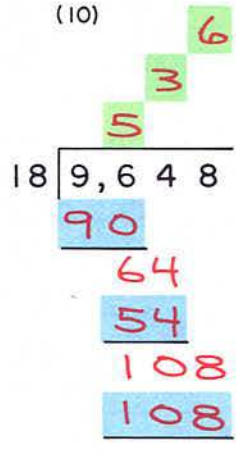
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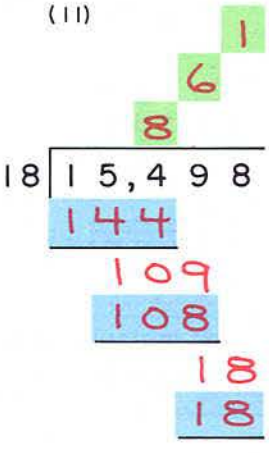
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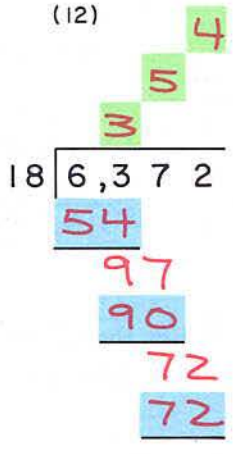
(7)


(8)


(9)


(10)


(11)


(12)


"My slogan is: 'When in doubt, don't leave the zeros out!' "

Tasks: *Pupils complete the lists of multiples of 18 and use the results to complete the twelve long division problems.*

Purpose: To focus the pupil's attention on the succession of subtractions that characterizes the long division algorithm.

Unifying Ideas: Structure; sets; number and counting; numeration; addition and subtraction; multiplication and division.

The Lesson: Each of the divisors in the examples on pupil page 42 is 18, and so the activity begins by making lists of all the multiplications that can be involved as long as the quotient is less than 1,000. Thus central attention is focused on the succession of subtractions which characterize the division algorithm.

Step by step, the zero digit that would normally be needed to maintain the place value of the nonzero digits is omitted. This is done simply as a matter of convenience leading to neater bookkeeping. The nonzero digits manage to maintain their place value because they are written in the appropriate columns with respect to the dividend.

The final step of the algorithm is to recopy or "bring down" digits from the dividend, one at a time, as they are needed.

Point out that whenever any uncertainty develops in an example, just don't leave out the zero digits.

Consider some exercises which call for reconstruction of division examples:

```
      74
18)1332
   126
    72
    72

      38
18)684
   54
   144
   144
```

The "partial dividends" correspond to the "partial products" in the usual long multiplication. For example, compare the first exercise above with the following:

```
  18
x 74
  72  ←
126   ←   } partial dividends
1332  ← total dividend
```

To suggest the extension of the long division algorithm, construct two more lists:

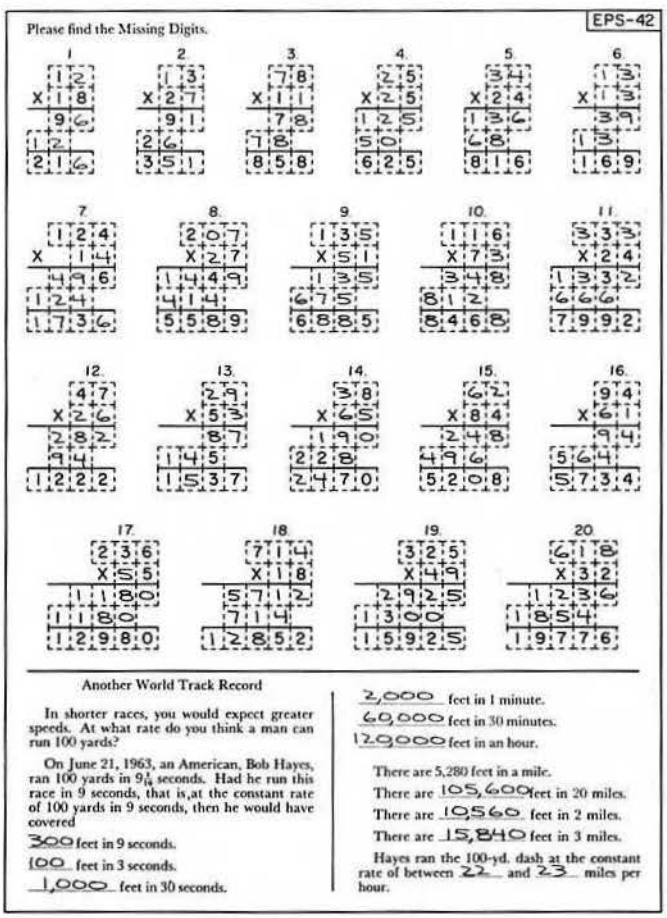
	0	1000	2000	3000	
18	0	18,000	36,000	54,000	etc.

	0	10,000	20,000	30,000	
18	0	180,000	360,000	540,000	etc.

and work out examples—both by the longer method and by the shorter method.

```
                1
               40
              300
             1000
            20000
18)384,138
   360000
    24138
    18000
     6138
     5400
      738
      720
       18
       18

                1
               4
              3
             1
            2
18)384,138
   36
    24
    18
     61
     54
      73
      72
       18
       18
```



"At this point," Martin explained, "when I stopped writing so many zeros, I realized that list C was the only list I needed for my work.

"Here are some examples using just one 37-list."

	0	1	2	3	4	5	6	7	8	9
37	0	37	74	111	148	185	222	259	296	333

a.

```
   23
37)851
   74
   111
   111
```

b.

```
    18
37)2,997
   296
     37
     37
```

c.

```
    52
37)1,924
   185
     74
     74
```

d.

```
    11
37)407
   37
    37
    37
```

e.

```
   128
37)4,736
   37
   103
    74
    296
    296
```

f.

```
   762
37)28,194
   259
    229
    222
      74
      74
```

g.

```
   530
37)19,610
   185
    111
    111
```

h.

```
   408
37)15,096
   148
    29
    00
    296
    296
```

"I improved the shorthand as shown in these examples."

i.

```
    74
37)2,738
   259
    148
    148
```

j.

```
    365
37)13,505
   111
    240
    222
     185
     185
```

k.

```
    942
37)34,854
   333
    155
    148
      74
      74
```

l.

```
    181
37)6,697
   37
   299
   296
     37
     37
```

m.

```
   256
37)9,472
   74
   207
   185
    222
    222
```

n.

```
   481
37)17,797
   148
    299
    296
      37
      37
```

o.

```
    69
37)2,553
   222
    333
    333
```

p.

```
   307
37)11,359
   111
    25
    00
    259
    259
```

Notes on Pupil Page 43

Tasks: Pupils complete the list of multiples of 37 and use the result to complete sixteen long division problems.

Purpose: To introduce the conventional way of writing the quotient in a long division problem. To provide practice in neat bookkeeping, which is essential to an efficient long division algorithm.

Unifying Ideas: Structure; sets; number and counting; numeration; addition and subtraction; multiplication and division.

The Lesson: The usual way of writing the quotient (or missing factor), anticipated in the preceding pages, is introduced on pupil page 43. Further, only one list of multiplications is completed before the work begins. Check now to make sure that pupils are fully aware that

if $37 \times 6 = 222$
then $37 \times 60 = 2220$ and
$37 \times 600 = 22200$, etc.

and that the shorthand developed eliminates the writing of the zero digits anyway.

The shorthand we are using depends on neat bookkeeping. To emphasize this aspect of the algorithm, we provide a space for each digit to be written. For the next few pages, we give the horizontal lines indicating the subtractions. Later, we provide only vertical guide lines.

Introduce the idea of checking the division by multiplication:

If $13{,}505 \div 37 = 365$
then

$$\begin{array}{r} 365 \\ \times\ 37 \\ \hline 2555 \\ 1095 \\ \hline 13{,}505 \end{array} \quad \text{and} \quad \begin{array}{r} 37 \\ \times\ 365 \\ \hline 185 \\ 222 \\ 111 \\ \hline 13{,}505 \end{array}$$

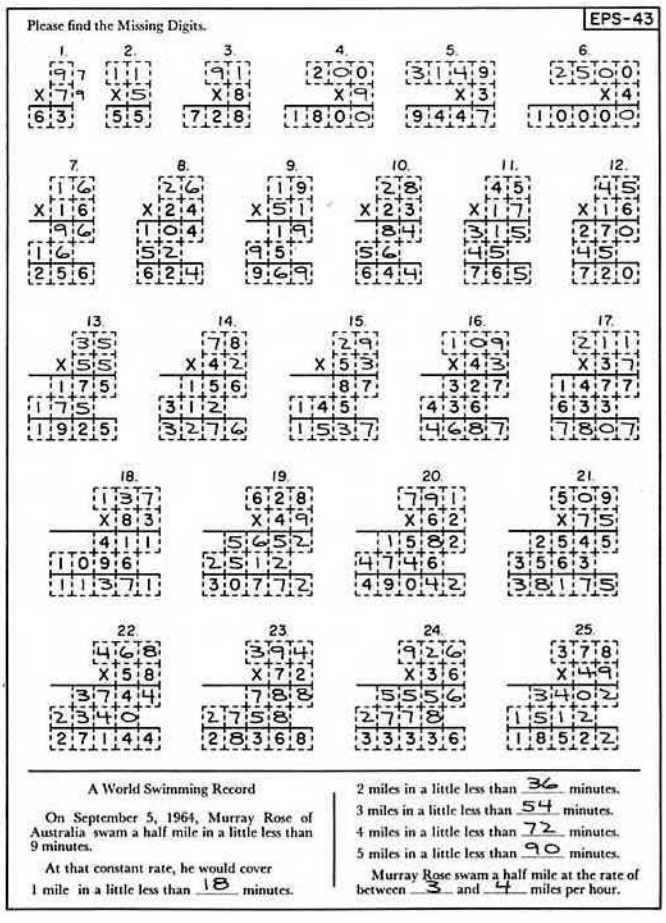
Notice that the long multiplication example on the right could be carried out by referring to the 37-list.

This makes its relationship to the long division example more obvious:

$$\begin{array}{r} 365 \\ 37\overline{)13{,}505} \\ 111 \leftarrow \\ \hline 240 \\ 222 \leftarrow \\ \hline 185 \\ 185 \leftarrow \\ \hline \end{array} \quad \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} \text{partial products}$$

And, again, reconstructions help develop a firmer grasp on the algorithm.

$$\begin{array}{r} 181 \\ 37\overline{)6697} \\ 37 \\ \hline 299 \\ 296 \\ \hline 37 \\ 37 \\ \hline \end{array}$$



"For the examples below," Martin explained, "I prepared several lists and used the one I needed for each example."

Below are three of Martin's lists.

	0	1	2	3	4	5	6	7	8	9
15	0	15	30	45	60	75	90	105	120	135
49	0	49	98	147	196	245	294	343	392	441
62	0	62	124	186	248	310	372	434	496	558

I.

```
   51
15|765
  75
  ---
   15
   15
  ---
```

II.

```
   21
49|1029
  98
  ---
   49
   49
  ---
```

III.

```
   87
62|5394
  496
  ---
   434
   434
  ---
```

IV.

```
   58
49|2842
  245
  ---
   392
   392
  ---
```

"Notice my shortcut when I use zero from one of my lists."

V.

```
   260
62|16120
  124
  ---
   372
   372
  ---
```

VI.

```
   308
49|15092
  147
  ---
   392
   392
  ---
```

VII.

```
   370
15|5550
  45
  ---
   105
   105
  ---
```

VIII.

```
   509
49|24941
  245
  ---
   441
   441
  ---
```

IX.

```
   704
49|34496
  343
  ---
   196
   196
  ---
```

X.

```
   280
62|17360
  124
  ---
   496
   496
  ---
```

XI.

```
   806
15|12090
  120
  ---
    90
    90
  ---
```

XII.

```
   903
49|44247
  441
  ---
   147
   147
  ---
```

VIII.

```
   378
62|23436
  186
  ---
   483
   434
  ---
    496
    496
  ---
```

XIV.

```
   965
15|14475
  135
  ---
    97
    90
  ---
     75
     75
  ---
```

XV.

```
   605
49|29645
  294
  ---
   245
   245
  ---
```

XVI.

```
   692
62|42904
  372
  ---
   570
   558
  ---
    124
    124
  ---
```

Tasks: Pupils complete lists of multiples of 15, 49, and 62 and use the results to complete ten long division problems.

Purpose: To provide practice with the conventional algorithm of long division, particularly with situations in which zeros occur in the quotient.

Unifying Ideas: Structure; sets; number and counting; numeration; addition and subtraction; multiplication and division.

The Lesson: Examples on pupil page 44 have divisors 15, 49, and 62. Thus three lists are prepared to facilitate the multiplication steps.

The final twelve examples (with a few exceptions) present details of the shorthand or record keeping when zeros occur in the quotient.

There are essentially only two ways in which zeros can occur in the quotient:

```
      260
62 ) 16,120
     124
     ---
      372
      372
      ---
        0
        0
```

?

```
      308
49 ) 15,092
     147
     ---
      (39)
      (00)
       392
       392
```

?

Are the steps indicated by parentheses necessary parts of the records? Clearly, the third step in the first example might well be omitted. However, the omission of the second step in the next example is a little less obvious. Tell pupils that "when in doubt, don't leave the zeros out."

Perhaps a few examples of short division may help strengthen the conviction that such steps can be omitted:

```
  309        107         50
2 ) 618    5 ) 535    8 ) 400    etc.
```

Further, we can point out the repetitions we make when we do record a subtraction of zero:

```
      509
49 ) 24,941
     245
      (44)
       00
      (44)1
       441
```

```
      806
15 ) 12,090
     120
      (9)
       0
      (90)
       90
```

Also, show the following variations:

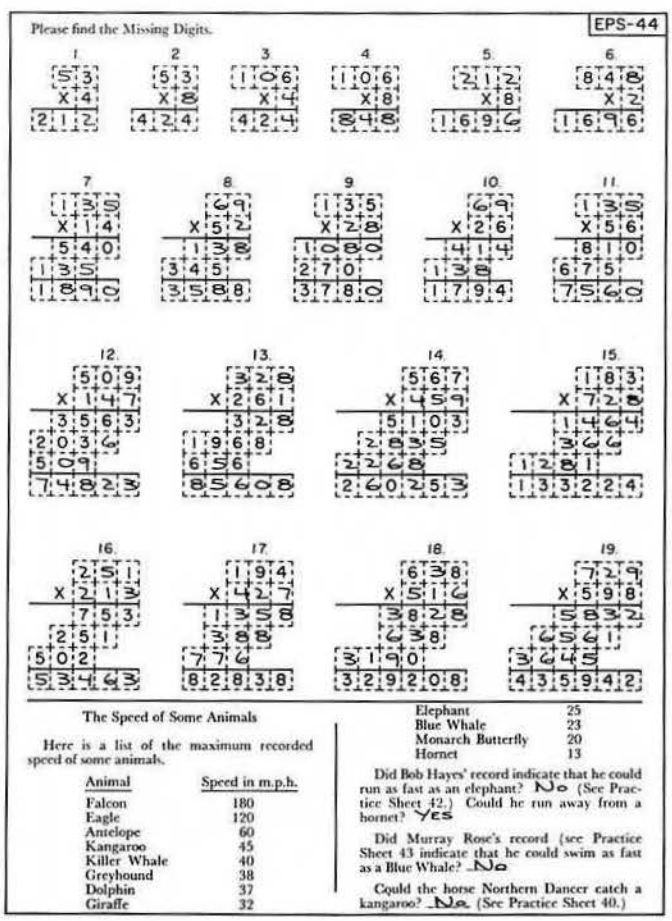
```
      2,009
62 ) 124,558
     124
     ---
       558
       558
       ---
```

and

```
      50,807
49 ) 2,489,543
     245
     ---
       395
       392
       ---
         343
         343
         ---
```

```
      500
25 ) 12,500
     125
     ---
```

Still, when in doubt, don't leave the zeros out.

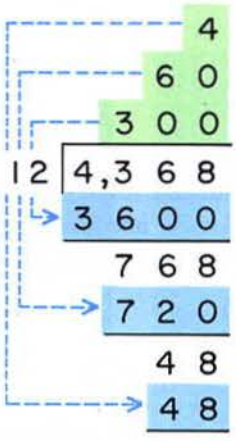


"Suppose you forget or lose your lists? What do you do then?" Carol asked.

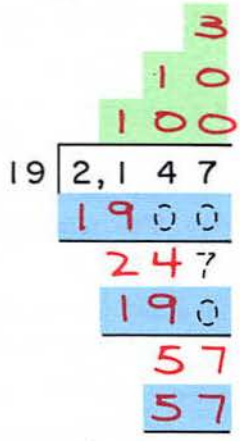
"Oh, I just multiply as I go along," Martin began. "I'll show you what my work looks like."

(*Please fill in the missing digits.*)

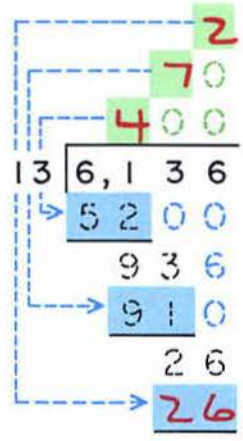
1.



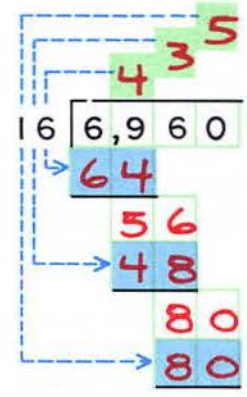
2.



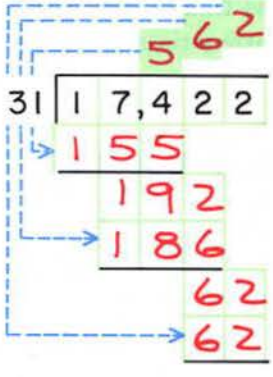
3.



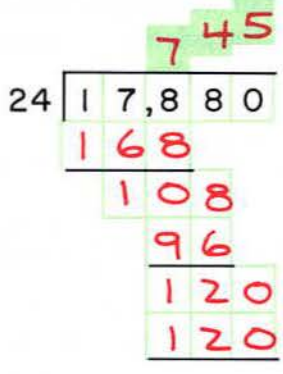
4.



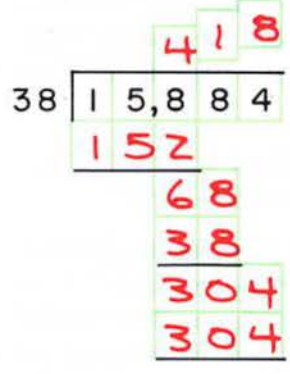
5.



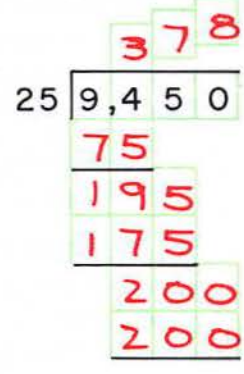
6.



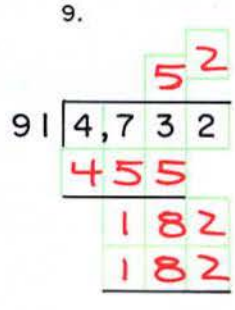
7.



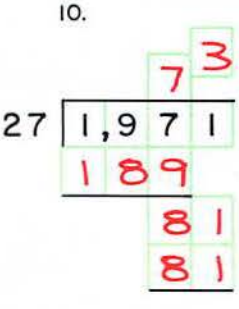
8.



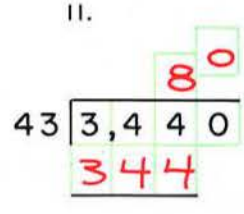
9.



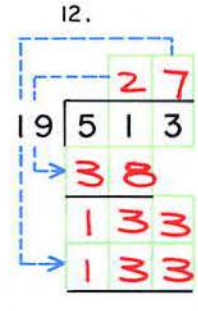
10.



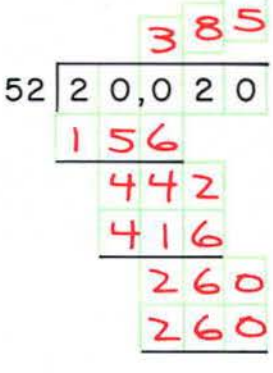
11.



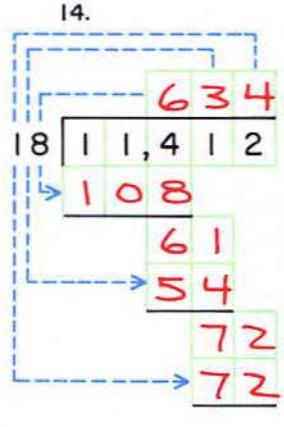
12.



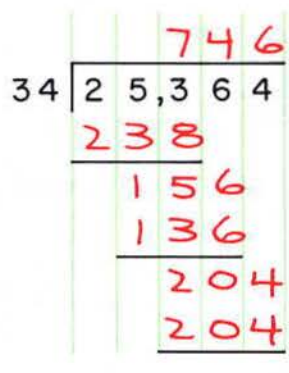
13.



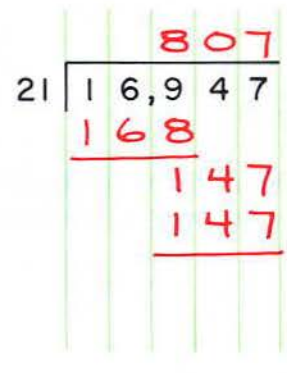
14.



15.



16.



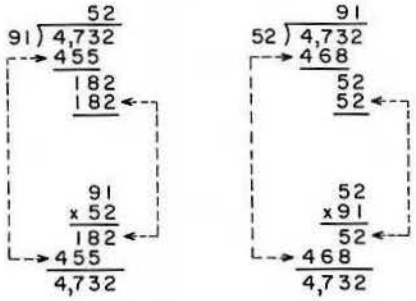
Tasks: *Pupils complete the sixteen long division problems.*

Purpose: To provide practice in the neat bookkeeping required in the usual long division algorithm.

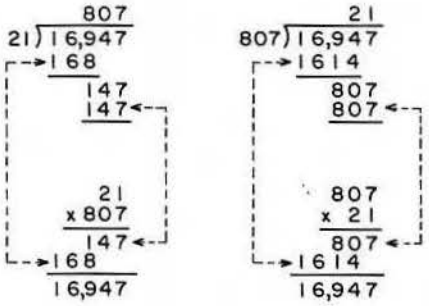
Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division.

The Lesson: The lists disappear. The multiplications must now be integrated with the subtractions. The first example recalls the origins of the shorthand we have been developing. The next few examples present various improvements in the record keeping. By the end of the page, pupils are asked to work without any crutch other than the vertical green lines, which are reminders of the importance of neat bookkeeping in the algorithm.

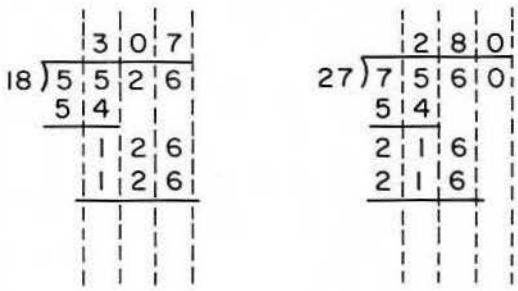
Practice sheet 45 focuses on the relationship between division and multiplication—one as a check on the other. You may find it helpful to look at some of the examples on the pupil page in this way. For example, exercise 9:



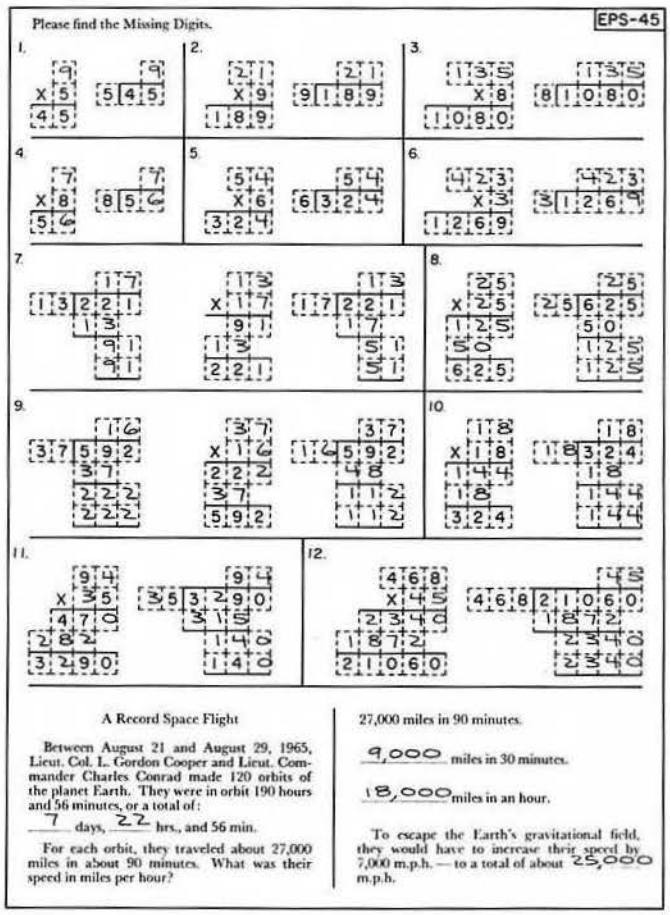
Example 16 on pupil page 45 will point out the way zeros may arise in the quotient and will also anticipate division by numbers of more than two digits:



Finally, reconstructions will provide still another vantage point from which to look at this algorithm.



Encourage pupils to make up their own examples for others to reconstruct. (They may reach out for really difficult ones—as a challenge.)



"Here are some lists I made for solving problems involving larger numbers."

P.	
X	127
0	0
1	127
2	254
3	381
4	508
5	635
6	762
7	889
8	1,016
9	1,143

Q.	
X	127
0	0
10	1,270
20	2,540
30	3,810
40	5,080
50	6,350
60	7,620
70	8,890
80	10,160
90	11,430

R.	
X	127
0	0
100	12,700
200	25,400
300	38,100
400	50,800
500	63,500
600	76,200
700	88,900
800	101,600
900	114,300

S.	
X	127
0	0
1000	127,000
2000	254,000
3000	381,000
4000	508,000
5000	635,000
6000	762,000
7000	889,000
8000	1,016,000
9000	1,143,000

T.	
X	127
0	0
10,000	1,270,000
20,000	2,540,000
30,000	3,810,000
40,000	5,080,000
50,000	6,350,000
60,000	7,620,000
70,000	8,890,000
80,000	10,160,000
90,000	11,430,000

"Suppose that I wanted to solve the following problems."

1.
$$127 \overline{)1,577,467} = 12,421$$

2.
$$127 \overline{)1,011,809} = 7,967$$

3.
$$127 \overline{)5,497,195} = 43,285$$

"I would select several numbers — one from list P, one from list Q, one from list R, and so on — until their sum is the dividend."

(*Please find the missing entries.*)

T.	1,270,000	10,000
S.	254,000	2,000
R.	50,800	400
Q.	2,540	20
P.	127	1
	1,577,467	12,421

0	0
889,000	7,000
114,300	900
7,620	60
889	7
1,011,809	7,967

5,080,000	40,000
381,000	3,000
25,400	200
10,160	80
635	5
5,497,195	43,285

Here are the same problems — with only the first steps — started in different shorthands. Please supply the missing entries.

4.
$$127 \overline{)1,577,467}$$
2,000
10,000
1270000
307467
254000

5.
$$127 \overline{)1,011,809}$$
7967
889
1228
1143
850

6.
$$127 \overline{)5,497,195}$$
43,285
508
417
381
361

asks: *Pupils complete the lists of multiples of 127 and use the results to complete long division problems.*

Purpose: To extend the long division algorithm.

Unifying Ideas: Structure; sets; number and counting; numeration; addition and subtraction; multiplication and division.

The Lesson: Again, because we wish the main attention to be on extending the algorithm, we return to lists—of multiples of 127.

During the first 127 working days of the year, a printer wished to examine the output of one of his presses. According to a counter on the press, 1,577,467 sheets had been printed in 127 days. What is the average number of sheets printed daily? From the lists, we notice that

10,000 a day for 127 days is 1,270,000
20,000 a day for 127 days is 2,540,000

So this inspection of our lists tells us that the average is more than 10,000 a day and less than 20,000 a day—and closer to the smaller figure. Since 10,000 a day accounts for 1,270,000 sheets, we have cut the problem down considerably:

$$\begin{array}{r} 1,577,467 \\ -\ 1,270,000 \\ \hline 307,467 \end{array}$$

In addition to 10,000 a day, the press turned out 307,467 sheets in 127 days. Back to the lists:

2,000 a day for 127 days is 254,000
3,000 a day for 127 days is 381,000

So an additional 2,000 sheets a day accounts for 254,000 of the 307,467 sheets:

$$\begin{array}{r} 307,467 \\ -\ 254,000 \\ \hline 53,467 \end{array}$$

Thus 12,000 sheets (10,000+2,000) would account for all but 53,467 sheets.

We notice that 400 a day for 127 days would account for 50,800 of these:

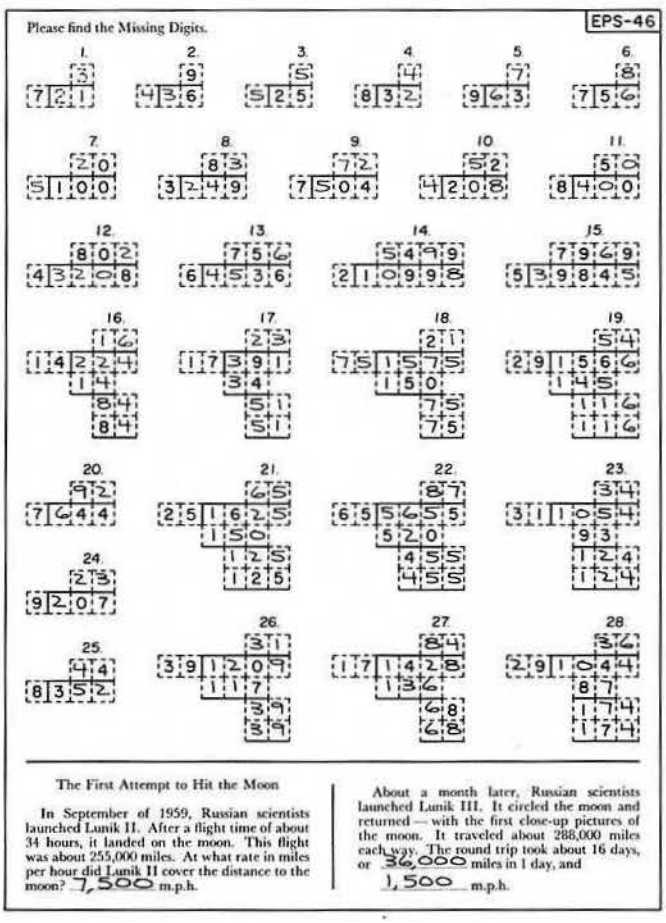
$$\begin{array}{r} 53,467 \\ -\ 50,800 \\ \hline 2,667 \end{array}$$

So all but 2,667 would be accounted for by 12,400 a day. The remaining 2,667 can be accounted for as follows:

$$\begin{array}{rl} \text{20 a day for 127 days is} & 2540 \\ \text{1 a day for 127 days is} & \underline{\ 127} \\ & 2667 \end{array}$$

Thus all 1,577,467 could have been produced in 127 days at the rate of 12,421 per day.

We encourage you to discuss examples in similar detail with pupils who have less than a firm grasp on the relationship between arithmetic and events in the real world.



In the following examples, just record the entries for the first step — as the first example indicates.

a.

```
      3
127 | 453,644
      381
```

b.

```
      6
127 | 8,143,113
      762
```

c.

```
      9
127 | 1,253,490
      1143
```

d.

```
      1
127 | 2,413
      127
```

e.

```
      9
127 | 124,173
      1143
```

f.

```
      7
127 | 9,867,910,160
      889
```

In the following examples, record the entries for the first two steps.

g.

```
      52
127 | 66,929
      635
       342
       254
```

h.

```
      19
127 | 253,873
      127
      1268
      1143
```

i.

```
      98
127 | 125,603,254
      1143
       1130
       1016
```

j.

```
      709
127 | 90,043
      889
       1143
       1143
```

k.

```
      66
127 | 8,382
      762
       762
       762
```

l.

```
      78
127 | 9,906
      889
      1016
      1016
```

m.

```
       9
127 | 1,143
      1143
```

Complete the following examples:

n.

```
      12421
127 | 1,577,467
      127
       307
       254
        534
        508
         266
         254
          127
          127
```

o.

```
       7967
127 | 1,011,809
       889
       1228
       1143
         850
         762
          889
          889
```

p.

```
       3518
127 | 446,786
      381
       657
       635
        228
        127
        1016
        1016
```

When in doubt, don't leave the zeros out!

Tasks: *Pupils use the lists of multiples of 127 on the previous page to complete one or more steps of the fifteen long division problems.*

Purpose: To extend the long division algorithm.

Unifying Ideas: Number and counting; numeration; addition and subtraction; multiplication and division.

The Lesson: Consider various possibilities for exercise *a*:

(1)
```
    2
127)453,644
  254
  1996
```

(2)
```
    4
127)453,644
  508
    ? 6
```

(3)
```
    9
127)453,644
 1143
 33934
```

(4)
```
    3
127)453,644
  381
   726
```

Because of the position of these digits in the quotient, we can consider them shorthand for 2,000, 4,000, 900, and 3,000, respectively. The first is inappropriate because it does not cut the problem down in size as much as it could. The second shows that the quotient must be less than 4,000. The third has the same fault as the first. Only the fourth gets us properly under way.

Each of these beginnings could be saved, but with considerable extra work and possible confusion. How might one proceed in each case? Here are a few suggestions, but encourage pupils to propose their own.

(1)
```
    1,572
    2,000   } 3,572
127)453,644
  254
  199
  127
   726
   635
    914
    889
     254
     254
```

(2)
```
      4,000
127)453,644
  508000
```

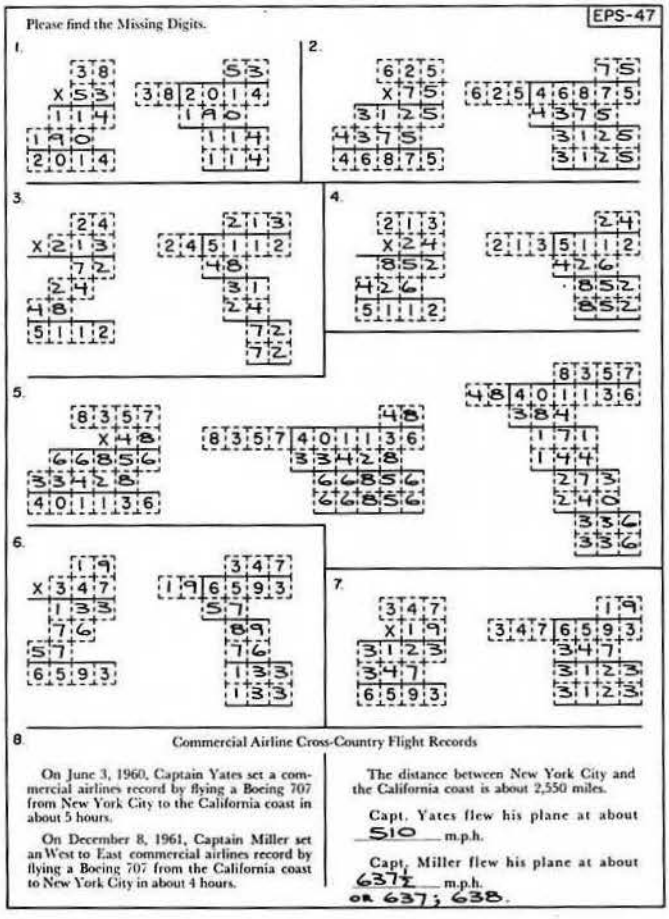
```
4,000
-428
3,572
```

```
      428
127)54,356
  508
   355
   254
   1016
   1016
```

(3)
```
    2,672   } 3,572
      900
127)453,644
 1143
 3393
  254
   853
   762
    914
    889
     254
     254
```

(4)
```
      3,572
127)453,644
  381
   726
   635
    914
    889
     254
     254
```

Such techniques are not presented for mastery, but rather for any value that a new point of view may have. For some, seeing that the same quotient results in every case may clarify the whole process. For others, the ability to build on a "bad" start is very convincing evidence of thorough understanding of the division algorithm.



Here are several of Martin's lists. Please fill in the missing entries.

X	0	1	2	3	4	5	6	7	8	9
19	0	19	38	57	76	95	114	133	152	171
43	0	43	86	129	172	215	258	301	344	387
65	0	65	130	195	260	325	390	455	520	585
209	0	209	418	627	836	1,045	1,254	1,463	1,672	1,881
758	0	758	1,516	2,274	3,032	3,790	4,548	5,306	6,064	6,822

If you wish, you can use the tables to help complete some of the examples on this page and the next page.

(1)
```
      5006
43 | 215,258
     215
     ----
       258
       258
       ---
```

(2)
```
      2300
65 | 149,500
     130
      195
      195
```

(3)
```
      9817
19 | 186,523
     171
      155
      152
       32
       19
       13
       13
```

(4)
```
       4
38 | 152
     152
```

(5)
```
      301
209 | 62,909
      627
       209
       209
```

(6)
```
      407
43 | 17,501
     172
       301
       301
```

(7)
```
      952
19 | 18,088
     171
       98
       95
        38
        38
```

(8)
```
       41
758 | 31,078
     3032
      758
      758
```

(9)
```
      9181
209 | 1,918,829
     1881
      378
      209
     1692
     1672
      209
      209
```

(10)
```
       31
22 | 682
     66
      22
      22
```

(11)
```
      206
65 | 13,390
     130
      390
      390
```

(12)
```
      8329
43 | 358,147
     344
      141
      129
       124
        86
        387
        387
```

(13)
```
      174
758 | 131,892
      758
     5609
     5306
      3032
      3032
```

Tasks: Pupils complete the list of multiples for five numbers and use the results to complete thirteen long division problems.

Purpose: To extend the long division algorithm.

Unifying Ideas: Structure; sets; addition and subtraction; multiplication and division.

The Lesson: Pupil pages 48 and 49 may be used to evaluate understanding of the usual long division algorithm (when remainders are not involved). Since the difficult computations for many of the exercises are done separately at the beginning, it should be easy to tell when the pupil is having computational difficulties and when he is having conceptual difficulties.

Again, more involved reconstructions may provide deeper insight.

$$\begin{array}{r} 63 \\ 42\overline{)2646} \\ 252 \\ \hline 126 \\ 126 \\ \hline \end{array} \qquad \begin{array}{r} 27 \\ 13\overline{)351} \\ 26 \\ \hline 91 \\ 91 \\ \hline \end{array}$$

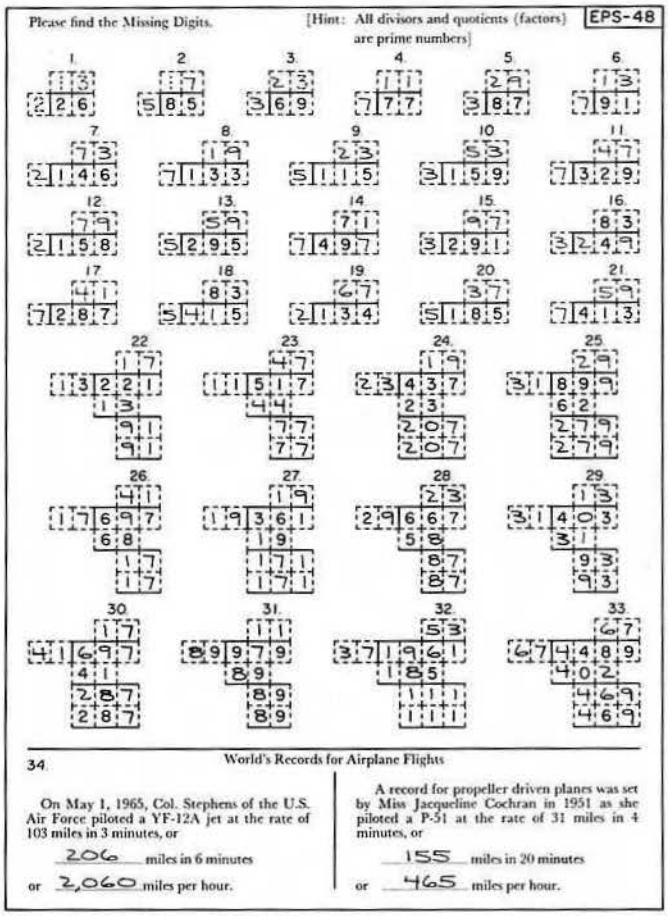
$$\begin{array}{r} 348 \\ 23\overline{)8,004} \\ 69 \\ \hline 110 \\ 92 \\ \hline 184 \\ 184 \\ \hline \end{array} \qquad \begin{array}{r} 293 \\ 36\overline{)10,548} \\ 72 \\ \hline 334 \\ 324 \\ \hline 108 \\ 108 \\ \hline \end{array}$$

$$\begin{array}{r} 317 \\ 124\overline{)39,308} \\ 372 \\ \hline 210 \\ 124 \\ \hline 868 \\ 868 \\ \hline \end{array} \qquad \begin{array}{r} 228 \\ 255\overline{)58,140} \\ 510 \\ \hline 714 \\ 510 \\ \hline 2040 \\ 2040 \\ \hline \end{array}$$

Some pupils may enjoy suggesting such reconstructions as challenges to the other pupils.

Practice sheet 48 is easier than it looks. The information that all divisors and quotients are prime numbers is a valuable key—and an essential one in exercises 4, 5, 6, and 16. If 1 (being nonprime) were not eliminated, it could be used as a divisor in exercises 4, 5, and 6. In exercise 16, $747 \div 9$ would work as well as $249 \div 3$. In each of the other exercises, the form assures a unique solution.

$$\begin{array}{r} 26 \\ 1\overline{)26} \\ \end{array} \quad \text{and} \quad \begin{array}{r} 13 \\ 2\overline{)26} \\ \end{array} \quad \text{etc.}$$



(14)
```
      76
15 | 1,140
     105
     ---
      90
      90
     ---
```

(15)
```
      34
29 | 986
     87
     ---
     116
     116
     ---
```

(16)
```
      119
81 | 9,639
     81
     ---
     153
      81
     ---
     729
     729
     ---
```

(17)
```
      2705
43 | 116,315
     86
     ---
     303
     301
     ---
       215
       215
       ---
```

(18)
```
      364
19 | 6,916
     57
     ---
     121
     114
     ---
       76
       76
       ---
```

(19)
```
      45
55 | 2,475
     220
     ---
       275
       275
       ---
```

(20)
```
      528
16 | 8,448
     80
     ---
      44
      32
     ---
      128
      128
     ---
```

(21)
```
      345
209 | 72,105
      627
      ---
       940
       836
       ---
       1045
       1045
       ----
```

(22)
```
      714
24 | 17,136
     168
     ---
      336
      240
      ---
       96
       96
       --
```

(23)
```
      70950
78 | 5,534,100
     546
     ---
       741
       702
       ---
        390
        390
        ---
```

(24)
```
      8462
25 | 211,550
     200
     ---
      115
      100
      ---
       155
       150
       ---
         50
         50
         --
```

(25)
```
      1090
37 | 40,330
     37
     --
      333
      333
      ---
```

(26)
```
      700
85 | 59,500
     595
     ---
```

(27)
```
      971
65 | 63,115
     585
     ---
      461
      455
      ---
        65
        65
        --
```

(28)
```
      958
120 | 114,960
     1080
     ----
       696
       600
       ---
        960
        960
        ---
```

(29)
```
      3517
758 | 2,665,886
      2274
      ----
       3918
       3790
       ----
        1288
         758
         ---
         5306
         5306
         ----
```

(30)
```
      780504
2 | 1,561,008
```

(31)
```
      12500
8 | 100,000
```

(32)
```
      1428572
7 | 10,000,004
```

Tasks: *Pupils complete nineteen long division problems.*

Purpose: To extend the long division algorithm.

Unifying Ideas: Structure; sets; addition and subtraction; multiplication and division.

The Lesson: As a variation on the work presented, some pupils may be interested in a method of changing some long division problems into a kind of chain of short divisions. To divide a number by 12, for example, one might elect to divide it by 2 and the result by 6, or divide by 3 and the result by 4.

$$\begin{array}{r} 13 \\ 12\overline{)156} \\ 12 \\ \hline 36 \\ 36 \\ \hline \end{array} \qquad \begin{array}{r} 13 \\ 6\overline{)78} \\ 2\overline{)156} \end{array} \qquad \begin{array}{r} 13 \\ 4\overline{)52} \\ 3\overline{)156} \end{array}$$

Consider several examples on pupil page 49:

$$15\overline{)1{,}140} \qquad \begin{array}{r} 76 \\ 3\overline{)228} \\ 5\overline{)1{,}140} \end{array} \qquad \begin{array}{r} 76 \\ 5\overline{)380} \\ 3\overline{)1{,}140} \end{array}$$

$$81\overline{)9{,}639} \qquad \begin{array}{r} 119 \\ 9\overline{)1{,}071} \\ 9\overline{)9{,}639} \end{array}$$

$$24\overline{)17{,}136} \qquad \begin{array}{r} 714 \\ 6\overline{)4{,}284} \\ 4\overline{)17{,}136} \end{array} \qquad \begin{array}{r} 714 \\ 3\overline{)2{,}142} \\ 8\overline{)17{,}136} \end{array}$$

$$25\overline{)211{,}550} \qquad \begin{array}{r} 8462 \\ 5\overline{)42{,}310} \\ 5\overline{)211{,}550} \end{array}$$

$$16\overline{)8{,}448} \qquad \begin{array}{r} 528 \\ 2\overline{)1{,}056} \\ 8\overline{)8{,}448} \end{array} \qquad \begin{array}{r} 528 \\ 8\overline{)4{,}224} \\ 2\overline{)8{,}448} \end{array}$$

$$\begin{array}{r} 528 \\ 4\overline{)2{,}112} \\ 4\overline{)8{,}448} \end{array} \qquad \begin{array}{r} 528 \\ 2\overline{)1{,}056} \\ 2\overline{)2{,}112} \\ 2\overline{)4{,}224} \\ 2\overline{)8{,}448} \end{array}$$

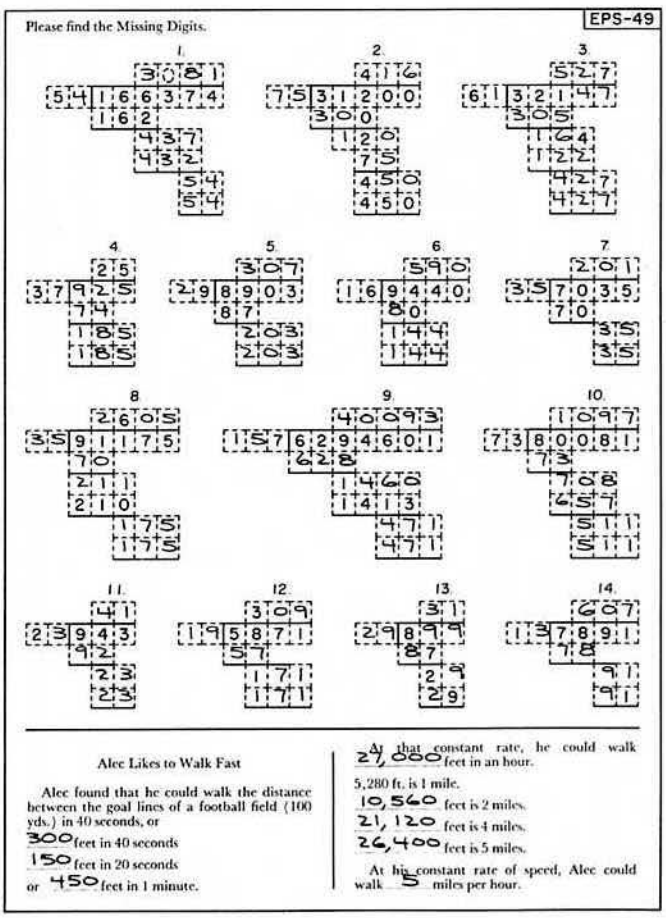
Any divisor that is a product of two or more single-digit numbers can be handled easily in this way. For example, since $126 = 7 \times 9 \times 2$, the quotient $413{,}784 \div 126$ can be found by a chain of short divisions, dividing by 7, 9, and 2 in any order.

$$\begin{array}{r} 3{,}284 \\ 7\overline{)22{,}988} \\ 9\overline{)206{,}892} \\ 2\overline{)413{,}784} \end{array} \qquad \begin{array}{r} 3{,}284 \\ 2\overline{)6{,}568} \\ 7\overline{)45{,}976} \\ 9\overline{)413{,}784} \end{array}$$

Could multiplication be handled in similar fashion? Since $168 = 3 \times 7 \times 8$, could we compute 596×168 by a chain of short multiplications?

$$\begin{array}{r} 596 \\ \times \quad 3 \\ \hline 1{,}788 \\ \times \quad 7 \\ \hline 12{,}516 \\ \times \quad 8 \\ \hline 100{,}128 \end{array}$$

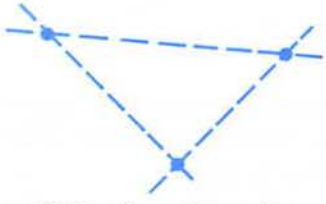
Let pupils explore this method further to discover whether it is reliable.



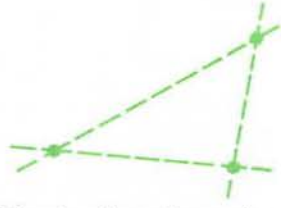
Surprises

"Here's a trick," Mary explained to Bill, "that has an interesting surprise to it. Mark three points on your piece of paper and draw a line through each pair of points. Then cut along those lines and a piece of the paper will drop out."

"Sure," said Bill, "and the pieces that drop out will be triangular regions."



Bill's triangular region



Mary's triangular region

"Now color each corner, and then tear the triangular region into three parts, a corner to each part.





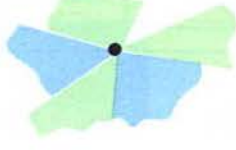
"Next," Mary continued, "I will paste one of my pieces with the colored corner point touching this dot I drew. Then you paste one of your pieces with its colored corner point on the dot I drew — and adjacent to the piece I pasted.



A corner of Mary's triangular region

A corner of Bill's triangular region

"Then I will paste another corner of my triangular region next to yours; you do the same; and I will paste my last corner down. And you paste your last corner down."

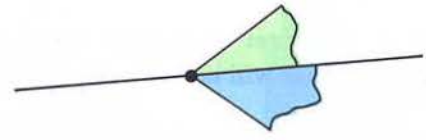


"It fits!" Bill exclaimed. "Will wonders never cease!"

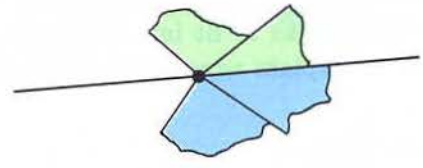
"Well," Mary said with a twinkle in her eye, "here is another surprise for you. Start out just as we did before. Draw any shape triangle you like, and so will I. Mark the corners on the triangular region and tear it so each piece contains one corner.

"This time, I'll mark a point on a line and paste one of my corners on one side of the line and you do the same on the other side."

Here is the result:



"Then I'll paste a second corner next to my first and you do the same on your side.

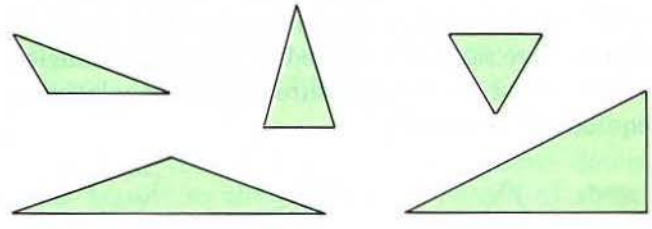


"Finally, I'll paste my last corner down and you do the same," Mary explained.

"Amazing!" Bill said. "Let's try it again, but this time I'm going to change the shape of my triangle."

What was the surprise?

Try it yourself. Cut out two triangular regions of any shape and repeat both of the experiments. Here are some shapes to consider:



Is there anything in particular to talk about in each of these triangular regions?

50 Surprises (The sum of the degree measures of the angles of a triangle is 180.)

Tasks: *Pupils draw triangles and cut out triangular regions. They tear off the three corners of each and paste them down with vertices and sides touching.*

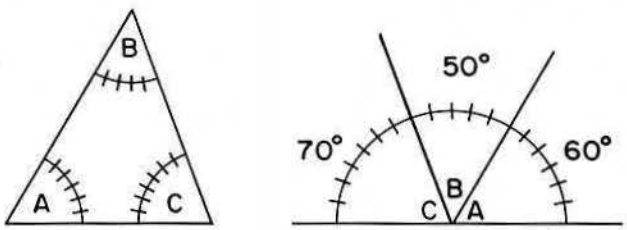
Purpose: To help pupils discover that the three angles of any triangle "fill" one side of a line; that is, that the sum of the degree-measures of the angles of a triangle is 180.

Unifying Ideas: Functions and relations; measurement; geometry.

The Lesson: Pupil pages 50 and 51 are intended to arouse the pupils' interest by presenting some surprising facts about triangles and four-sided figures (quadrilaterals).

We are, of course, not taking a highly structural approach to geometry. Rather we encourage pupils to make simple experiments with paper figures.

The chief idea of the experiments is pictured at the bottom of pupil page 50, where the three corners of a triangle are fitted together to "fill" one side of a line:



Analyzing the phenomenon numerically, we find that corner A covers 60 degrees of a 180-degree protractor, corner B covers 50 degrees, and corner C covers 70 degrees—180 degrees in all.

Mary's second trick on page 50 demonstrates the same fact. She, too, fills one side of a line with the corners of a triangle. Bill repeats the experiment by filling the other side of the line.

Mary's second trick explains her first trick. The corners of Mary's triangle cover 180 degrees; the corners of Bill's triangle also cover 180 degrees. So, the corners of both triangles should cover 360 degrees, which corresponds to filling both sides of the line. By mixing Bill's corners with her own, Mary demonstrates that the total space taken has nothing to do with the order in which the pieces are assembled.

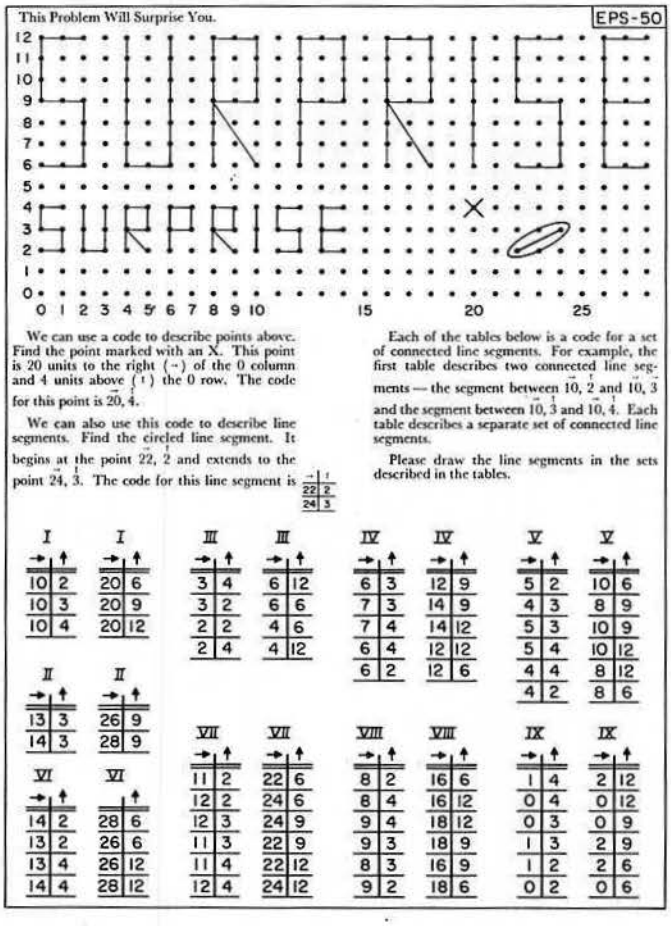
These points need not be emphasized in class. It is better to let these ideas occur gradually to the pupils than to offer a full explanation immediately.

What is a Triangle?

You might point out to pupils that a triangle has three *vertices* or "corners"; that it is constructed by drawing the three segments—called *sides*—joining the vertices; that the word *TRIangle* refers to the figure's three angles.

By looking at and talking about triangles, pupils will become aware of the common characteristics of all triangles and the special characteristics of some triangles. Rigorous definitions of terms is certainly premature at this time. Let pupils get a feeling about these shapes, using freely whatever language seems natural to them.

Practice sheet 50 provides experience in graphing and a surprise.



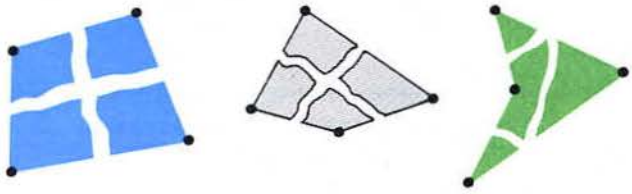
Bill's Surprises

The next day, Bill came to class with a surprise of his own.

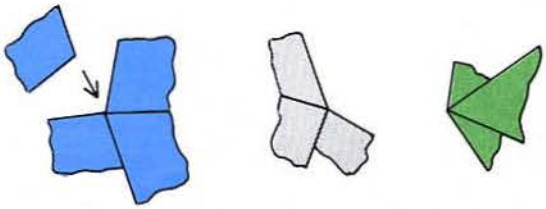
"Mary," he said, "I worked out a trick something like yours. Mark any four points and connect them so you have a four-sided figure. Like this:



"Now, cut along the line that connects the points. Next, tear each shape into four pieces so that each piece contains one of the corners — one of the original four points.



"Then, arrange each group of four pieces so that the corners touch at the same point, and so that their sides just touch.



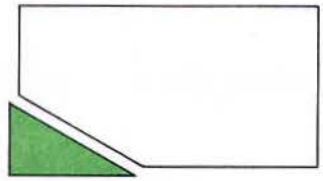
"Surprised?" Bill asked.

"I surely am surprised," Mary replied. "Would it work for squares, rectangles, and other four-sided shapes?"

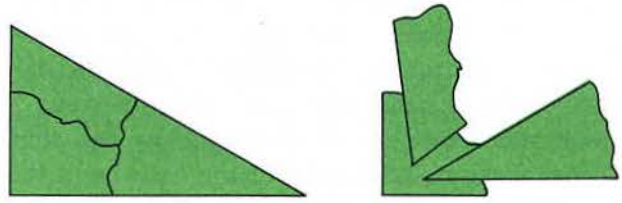
"Try it!"



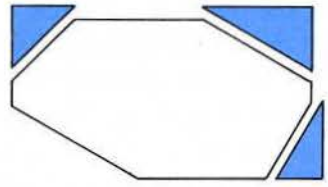
"The other fitting trick is this: Cut a triangular corner off a rectangular piece of paper.



"Tear the triangular corner into three pieces so that each piece has one of the three corners of the triangle. Now lay down the square corner and try to fit the other two corners into it."



"It's a 'fitting' surprise," Mary agreed. "I wonder if you could get the same results if you tried triangles of different shapes and sizes?"



"Try it," Bill said.

Tom had been watching and trying some experiments of his own.

"I think the same thing would happen," he said, "even if you turned all the pieces over or just some of them."

"That's hard to believe," Mary said.

"Try it, then," said Tom.

Make a study of your own of these fitting tricks. Try each trick with many different shapes. Turn some of the pieces over.

As you try each experiment, paste down your results so they become a record of these examples of 'fitting.'

Perhaps you can invent a 'fitting trick' of your own.

51 Bill's surprise (The sum of the degree measures of the angles of a quadrilateral is 360.)

Tasks: *Pupils draw and cut out four-sided figures and right triangles. They tear off the corners and paste them down with vertices and sides touching.*

Purpose: To help pupils discover that the four angles of any four-sided polygon "fill" both sides of a line; to help pupils discover that the two smaller angles of a right triangle together "fit" the right angle.

Unifying Ideas: Functions and relations; measurement; geometry.

The Lesson: The first column on pupil page 51 is related to facts that will be discovered on pupil page 56.

The second column deals with triangles that have one angle of 90 degrees. Since the three angles of a triangle must contain 180 degrees, any 90-degree angle must "take as much space" as the other two angles put together. This is indicated by showing how the "square corner" (right angle) is big enough for the other two angles to fit into it exactly. Do not feel obliged to explain why the tricks on this page work.

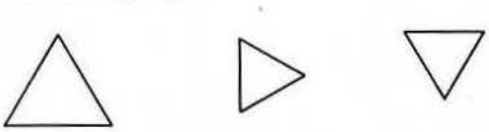
Kinds of Triangles

Pupils will be interested to know that triangles with certain characteristics have special names. When studying the next several pages, have pupils construct examples of these triangles on graph paper or with sticks, strips of paper, or wire.

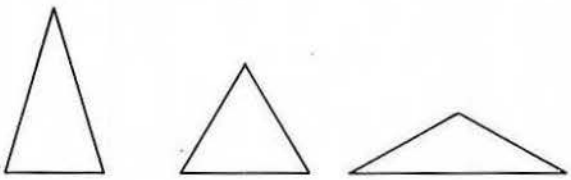
The Scalene Triangle. A scalene triangle is one in which the sides are all of different lengths.



The Equilateral Triangle. An equilateral triangle has three sides equal in length (equi = equal; lateral = sides). Equilateral triangles have angles equal in size and are sometimes called equiangular triangles. (What must be the measure of each angle in an equilateral triangle?)

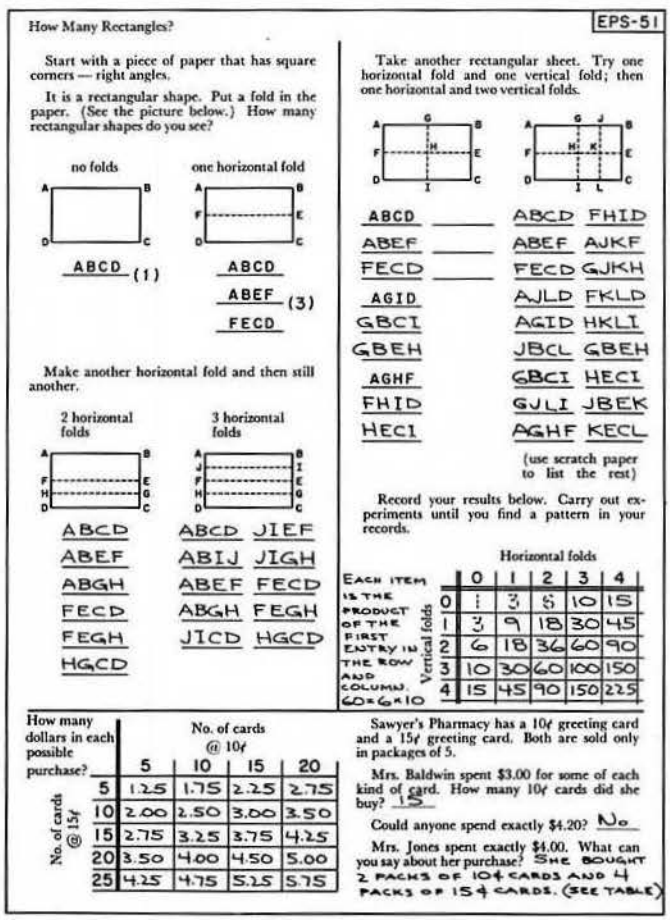


The Isosceles Triangle. An isosceles triangle has two sides of the same length. The third may differ in length and is called the base. The angles which contain the base are called the base angles and are equal in size. The remaining angle is opposite the base and is called the vertex angle.



In the notes on pupil page 53, triangles are classified according to the size of their largest angle. The classes are right, obtuse, and acute.

The top section of practice sheet 51 offers a problem which teams of pupils may wish to explore to see which team can first discover the pattern and complete the table without actually counting every rectangle.



How Many Rectangles? EPS-51

Start with a piece of paper that has square corners — right angles.

It is a rectangular shape. Put a fold in the paper. (See the picture below.) How many rectangular shapes do you see?

no folds — ABCD (1)

one horizontal fold — ABCD, ABEF, FECD (3)

Make another horizontal fold and then still another.

2 horizontal folds — ABCD, ABEF, ABGH, FECD, FEGH, HGCD

3 horizontal folds — ABCD, JIEF, ABIJ, JIGH, ABEF, FECD, ABGH, FEGH, JICD, HGCD

Take another rectangular sheet. Try one horizontal fold and one vertical fold; then one horizontal and two vertical folds.

ABCD	ABCD FHID
ABEF	ABEF AJKF
FECD	FECD GJKH
AGID	AJLD FKLD
GBCI	AGID HKLI
GBEH	JBCL GBEH
AGHF	GBCI HECI
FHID	GJLI JBEK
HECI	AGHF KECL

(use scratch paper to list the rest)

Record your results below. Carry out experiments until you find a pattern in your records.

Each item is the product of the first entry in the row and column. 60=6×10

Vertical folds \ Horizontal folds	0	1	2	3	4
0	1	3	6	10	15
1	3	9	18	30	45
2	6	18	36	60	90
3	10	30	60	100	150
4	15	45	90	150	225

How many dollars in each possible purchase?

No. of cards @ 15¢ \ No. of cards @ 10¢	5	10	15	20
5	1.25	1.75	2.25	2.75
10	2.00	2.50	3.00	3.50
15	2.75	3.25	3.75	4.25
20	3.50	4.00	4.50	5.00
25	4.25	4.75	5.25	5.75

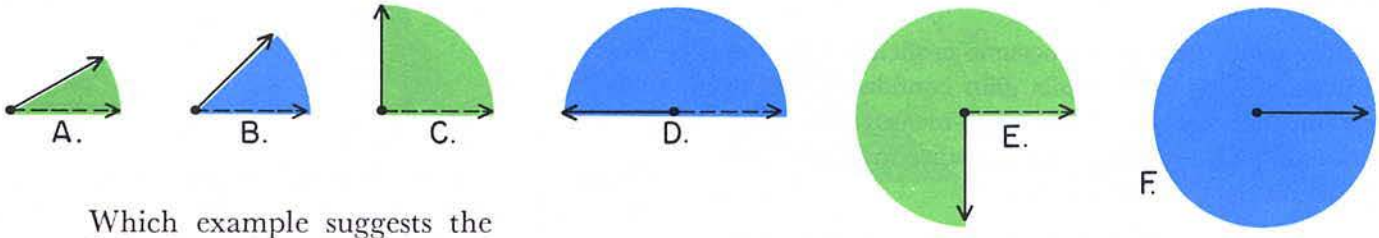
Sawyer's Pharmacy has a 10¢ greeting card and a 15¢ greeting card. Both are sold only in packages of 5.

Mrs. Baldwin spent $3.00 for some of each kind of card. How many 10¢ cards did she buy? 15

Could anyone spend exactly $4.20? No

Mrs. Jones spent exactly $4.00. What can you say about her purchase? She bought 2 packs of 10¢ cards and 4 packs of 15¢ cards. (See table)

Imagine a pointer or clock-hand that can be swung around a pivot. Before it has been swung, suppose it points to the right as shown by the broken line. Of the many positions into which it can turn, a few are shown.



Which example suggests the following amounts of turning?

smallest amount of turning A

half-way around D

one-fourth of the way around C

all the way around F

three-fourths of the way around E

half of a quarter of a full turn B

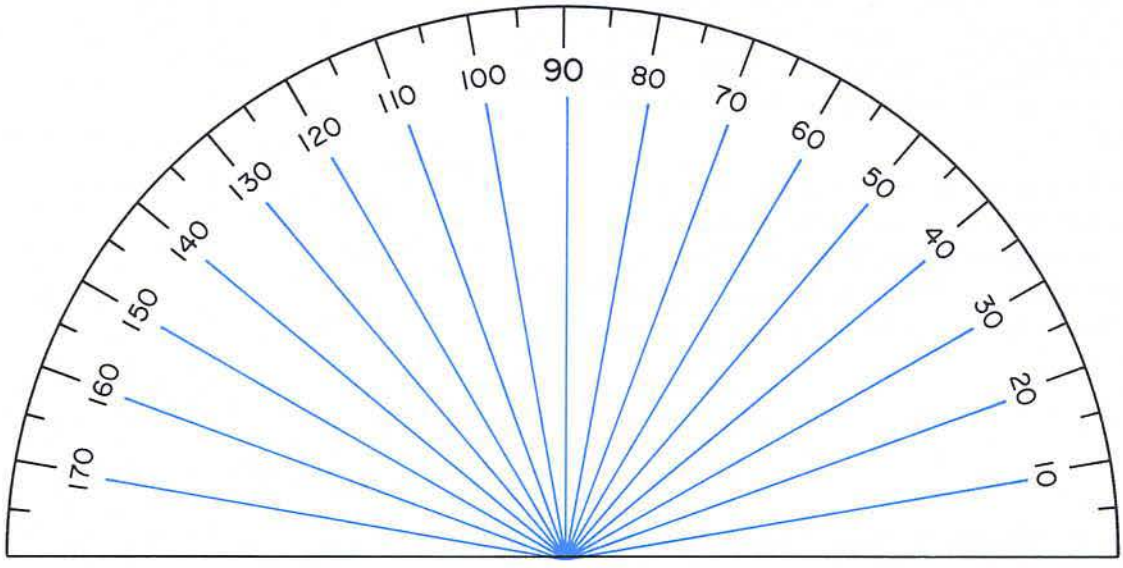
Mathematicians have agreed to consider a quarter of a full turn as a turn of 90 degrees — in shorthand, 90°.

Then, a half of a turn is 180°.

Three-fourths of a turn is 270°.

A full turn must be 360°.

An instrument such as the one below is called a PROTRACTOR. It is used to measure the amount of turning that would be required in making a particular angle.



The protractor is a valuable tool. It is used to measure the size of an angle (the amount of turning it would take for a hand to sweep from one side of an angle to the other). Cut out triangles of many different shapes and sizes. Mark the angles of each *a*, *b*, and *c*. Use the protractor to measure each angle. Record the results in columns III-XIII in the chart on page 53. What do you notice about the sum of the measures of each group of angles?

52 Swinging around a pivot (Investigating a protractor)

Tasks: *Pupils estimate the size of angles. They study the protractor.*

Purpose: To provide experiences in estimating the size of angles; to teach pupils to measure angles with a protractor.

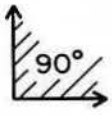
Unifying Ideas: Functions and relations; measurement; geometry.

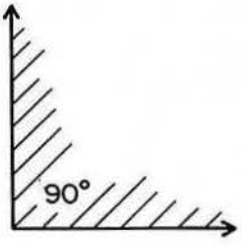
The Lesson: Pupil page 52 deals with angles and their measurement, beginning with the idea of "amount of turning." Have students imagine how much the hour hand of a clock turns between 12 o'clock and 1 o'clock. Then ask if they can think of other amounts of turning of the hour hand that would be equal to this. (4 o'clock to 5 o'clock.) What amounts would be equal to twice as much? (12 o'clock to 2 o'clock.)

A clock face is divided into 60 parts of equal size. A protractor, also divided into parts of equal size, is designed on the basis that a circle can be divided into 360 equal parts—360 degrees. This means that the distance between two minute marks on a circular clock face corresponds to 6 degrees on a protractor. In other words, between 12 o'clock and 1 o'clock, the hour hand turns through 30 degrees; and between 12 o'clock and 3 o'clock, the hour hand turns through 90 degrees.

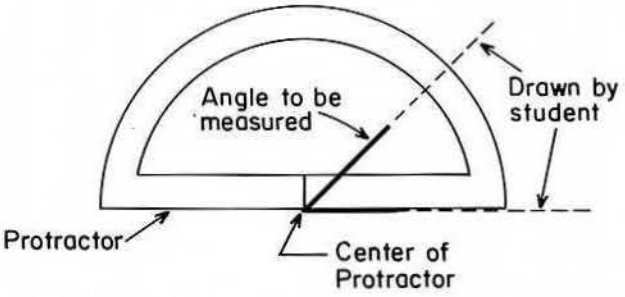
The abbreviation for 90 degrees is 90°, and pupils should become familiar with this notation from the beginning.

Pupils need to know that the size of an angle does not depend on the size of the object which "contains" the angle. The corner of a matchbox, of a piece of rectangular paper, of a house, of a football field, of a city block—each forms an angle of 90°.



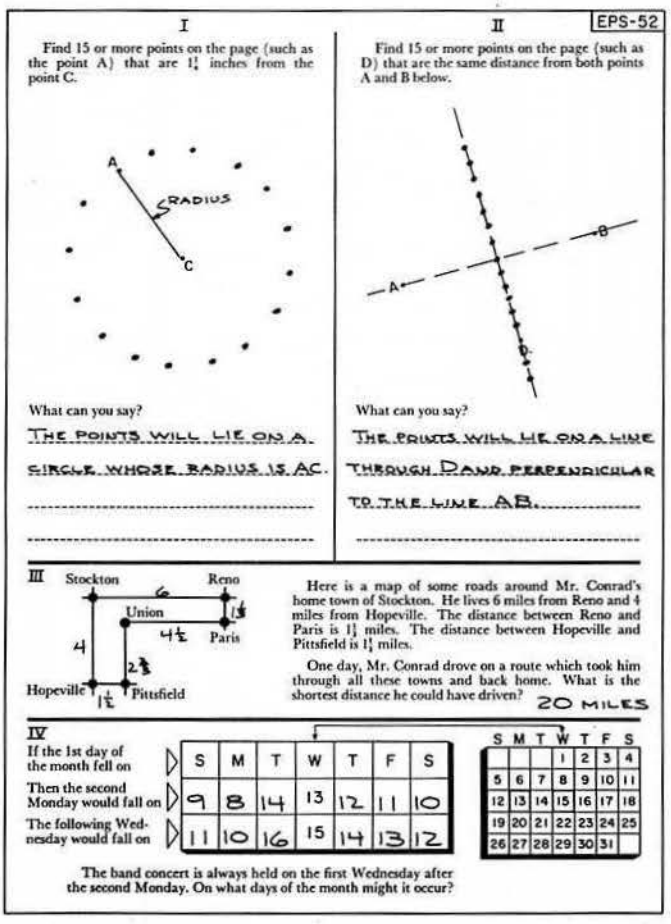


Pupils should have and use their own protractors. Very good ones can be bought for about a quarter. You may wish to discuss the scale pupils find on their protractors.

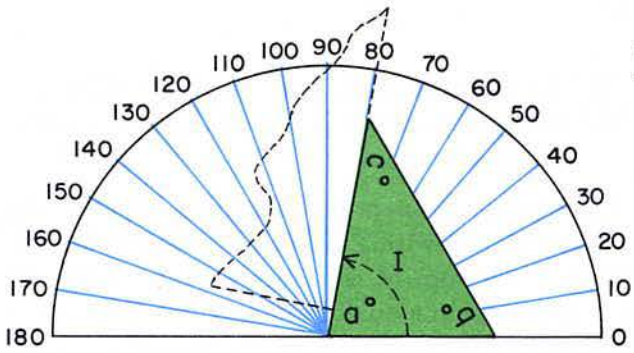


Pupils may find that the sides of an angle they wish to measure are drawn too short to reach the rim of the protractor. In such a case, they can lengthen the sides (as shown by the dotted lines in the diagram) or lay pieces of paper against the sides and measure the angle between the edges of the pieces of paper.

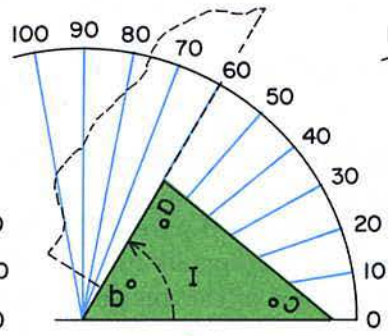
The four problems on practice sheet 52 may be explored by teams, who might explain their findings to the class.



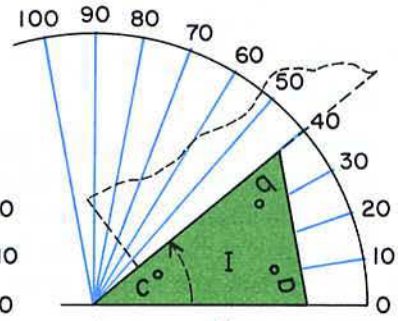
If the sides of a triangle do not intersect the protractor scale, place a piece of paper along the sides (as the dotted black lines suggest) so that the angles of the triangle can be measured.



a = 80



b = 60

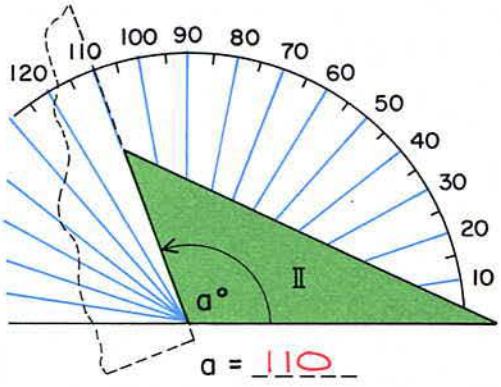


c = 40

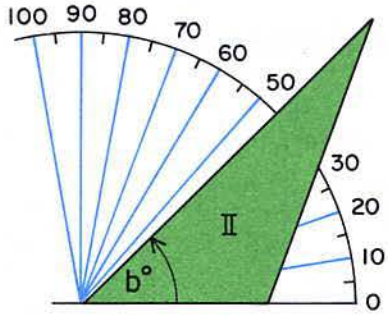
Examples:

Use the table below to record your experiments with triangles I and II.

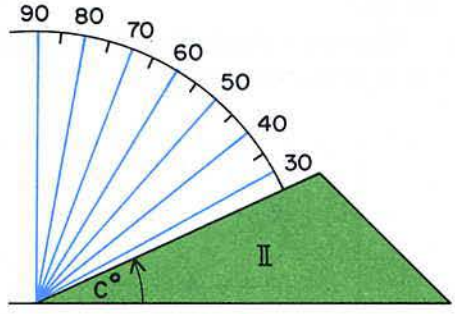
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII
a	80	110											
b	60	45											
c	40	25											
Total	180	180											



a = 110



b = 45

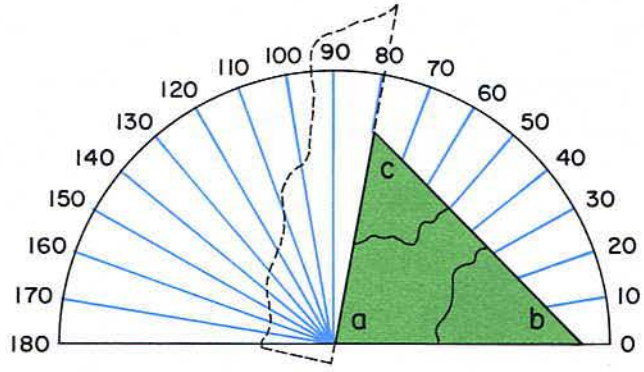


c = 25

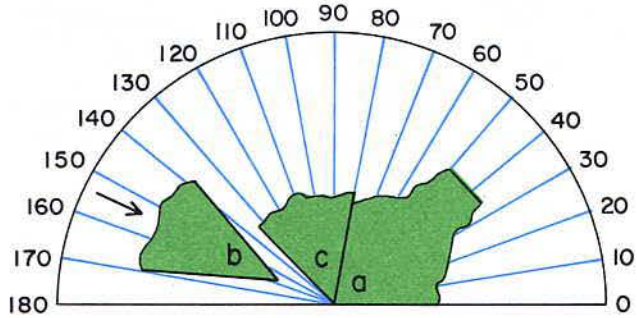
Conduct other experiments. Cut triangular shapes out of paper or cardboard. Place them on the protractor on page 52. Measure each angle and record the results for each triangle in columns III through XIII of the table.

If your measurements could be exact, what would you expect the sum of the degree measures of each group of three angles to be?

Use any triangular shape. Paste one corner down as if you were going to measure the angle, as shown below:



Tear off the other two corners. Fit one as shown. What will happen if the other corner is pushed into place?



53 Using the protractor to determine the sum of the degree measures of the angles of a triangle

Tasks: *Pupils cut out triangles and measure their angles with a protractor, keeping records of the results.*

Purpose: To help pupils become proficient with the protractor. To help pupils gain confidence in the notion that the sum of the angle-measures in every triangle is 180°. To help pupils realize that measurement involves approximation.

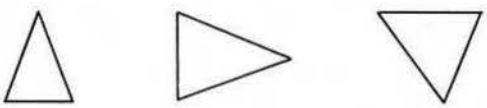
Unifying Ideas: Functions and relations; measurement; geometry.

The Lesson: The experiments on this pupil page are intended to show that the sum of the angle-measures of a triangle is 180°. Errors of measurement, however, are unavoidable, and if the totals come between 170° and 190°, this is as much as we can expect. Similarly, measurements of the same triangle by different pupils will produce different totals. Therefore we can hardly claim that these results *prove* that the total number of degrees in the angles of a triangle is exactly 180. You will have to assure pupils that assuming the total to be 180 will not cause difficulty in most practical problems.

Other Classes of Triangles

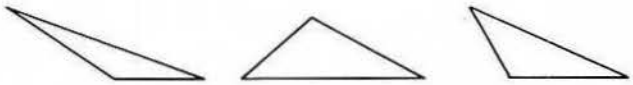
In the notes on pupil page 51, triangles were classified according to lengths of sides. The classes were scalene, equilateral, and isosceles. Another way of classifying triangles is according to the size of their largest angle.

The Acute Triangle. An acute angle is one whose measure is less than 90°. An acute triangle is one whose largest angle is acute. It follows, then, that no angle of an acute triangle is as large as a 90° angle. All three angles of an acute triangle are acute.

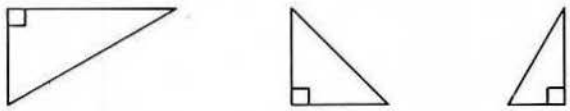


(Can a scalene triangle be acute? Can an isosceles triangle be acute? An equilateral triangle?)

The Obtuse Triangle. An obtuse angle is one whose measure is more than 90° but less than 180°. An obtuse triangle is a triangle whose largest angle is an obtuse angle. (Could a triangle have two obtuse angles? Can an isosceles triangle be obtuse?)

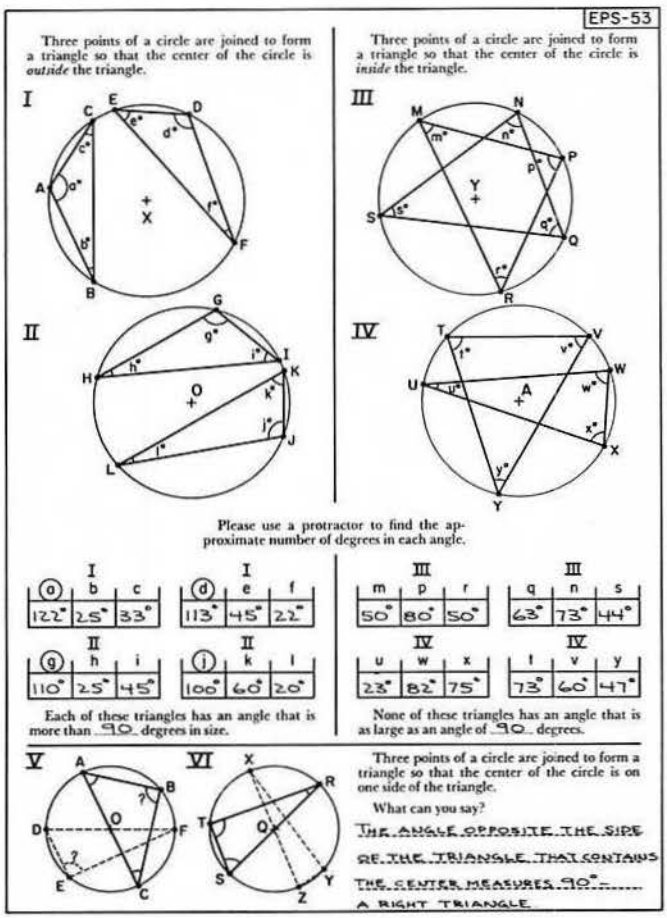


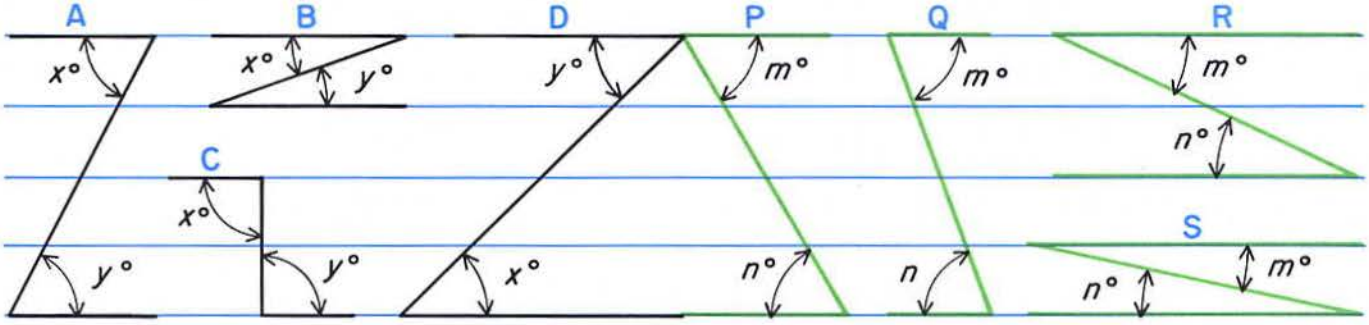
The Right Triangle. A right triangle is a triangle whose largest angle is an angle of 90°.



(Can a right triangle be isosceles? Can it be scalene? Can it be acute?)

Practice sheet 53 involves measuring and recording angles of acute, obtuse, and right triangles.





For each of the Z-shapes above, use a protractor to measure angles marked $x°$ and $y°$. For each of the S-shapes, measure angles marked $m°$ and $n°$.

On lined paper, draw some Z's of your own — of the regular and reversed varieties — and record similar measurements of the angles in the chart below.

Number of degrees	A	B	C	D	E	F	G	H	I	J	K
x	62°	20°	90°	45°							
y	62°	20°	90°	45°							

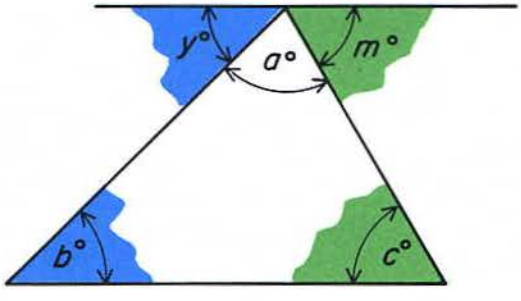
Number of degrees	P	Q	R	S	T	U	V	W	X	Y	Z
m	60°	70°	25°	12°							
n	60°	70°	25°	12°							

In shorthand, we can say very briefly what we note: $x = y$ and $m = n$. In the Z and the reverse Z, the angles are the same size.

We call this the "Z-result."

(Turn this page on the side and you might decide to call it the "N-result.")

Below is a copy of the shape or design that appears in the center at the top of the page:

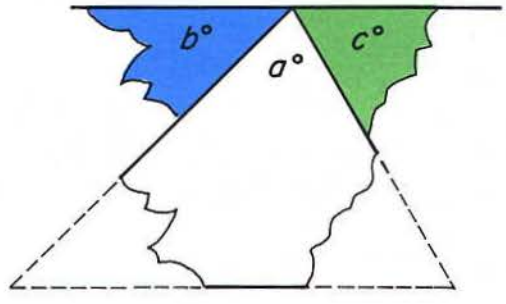


Only the labels have been changed, and a little color has been added.

You can see it as a Z and a S put together — or as a triangle with a line through one of the corners.

The Z-result indicates that $b = y$ and that $c = m$.

If we cut out the corners marked $b°$ and $c°$ and paste them on angles marked $y°$ and $m°$, we have:



54 Using the protractor to measure the angles of a Z-shape (If two parallel lines are cut by a transversal, then the alternate interior angles are equal.)

Tasks: *Pupils measure angles and record results in a table.*

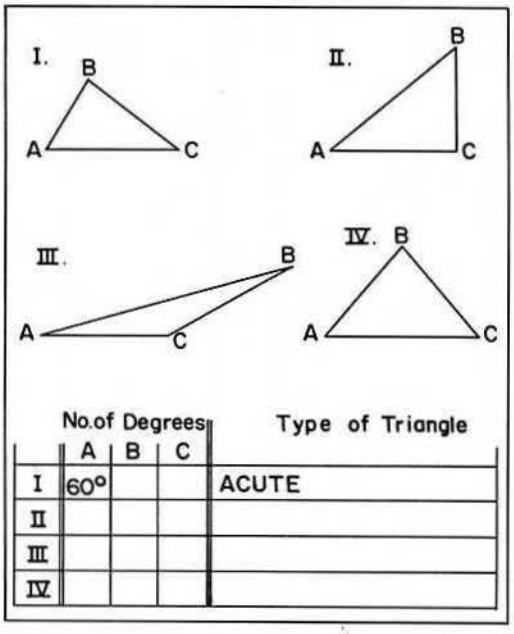
Purpose: To help pupils become more proficient with the protractor. To help pupils discover that alternate interior angles are equal in size if parallel lines are cut by a straight line. To help pupils further explore the sum of the measures of the angles of a triangle.

Unifying Ideas: Functions and relations; measurement; geometry.

The Lesson: As on pupil page 53, errors of measurement will occur. The tables can, at best, show that the two angles are about the same size. For everyday purposes, it has been found very satisfactory to assume that these angles are equal in size.

At the bottom of the page, an argument is given to show that a consequence of the "Z-result" is the fact that the sum of the angle measures of a triangle is 180°.

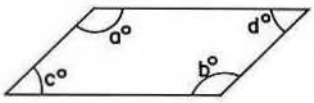
Prepare a sheet like the following and have pupils draw triangles as indicated:



	No. of Degrees A	B	C	Type of Triangle
I	60°			ACUTE
II				
III				
IV				

I. A scalene triangle with one angle of 60°.
II. A right triangle with one angle of 40°.
III. An obtuse triangle with an angle of 120°.
IV. An isosceles triangle with an 80° angle opposite the base.

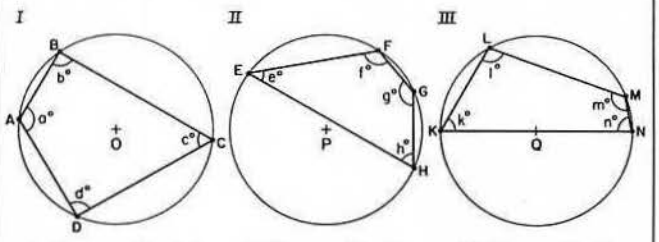
Practice sheet 54 provides further experiments with a protractor with a rather surprising result. We find that the sum of the degree-measures of the opposite angles in a quadrilateral is 180. This result will hold only for quadrilaterals inscribed in a circle. Have pupils experiment with other quadrilaterals—those that cannot be inscribed in a circle, such as a nonrectangular parallelogram; this ought to be sufficient to assure pupils that the sum of the measures of either pair of opposite angles is not 180°:



a + b > 180
c + d < 180

EPS-54

Use a protractor to measure the angles of each quadrilateral (four-sided figure).



I. The center of the circle is inside the quadrilateral.

a	b	c	d
120°	90°	60°	90°

a + b = 210° c + d = 150°
a + c = 180° b + d = 180°
a + b + c + d = 360°

II. The center of the circle is outside the quadrilateral.

e	f	g	h
40°	120°	140°	60°

e + f = 160° g + h = 200°
e + g = 180° f + h = 180°
e + f + g + h = 360°

III. The center of the circle is on the quadrilateral.

k	l	m	n
60°	100°	120°	80°

k + l = 160° m + n = 200°
k + m = 180° l + n = 180°
k + l + m + n = 360°

What can you say? FOR QUADRILATERALS WHOSE VERTICES ARE ON A CIRCLE, THE SUM OF THE DEGREE-MEASURES OF A PAIR OF OPPOSITE ANGLES (LIKE THE ANGLES AT A AND C) IS 180; THE SUM OF THE DEGREE-MEASURES OF ALL FOUR ANGLES IS 360.

IV

I am thinking of a fraction. The denominator is not less than 4 nor more than 8. The numerator is not more than 40 nor less than 20.

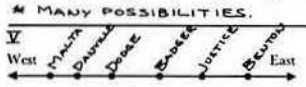
Write fractions I could be thinking of. Also indicate the largest and smallest fractions I might be thinking about — in their reduced form.

Largest Smallest

$\frac{25}{4}$ $\frac{30}{6}$ $\frac{35}{8}$ $\frac{40}{7}$ $\frac{40}{5}$ $\frac{40}{8}$ $\frac{20}{4}$ $\frac{28}{4}$ $\frac{20}{8}$ $\frac{37}{8}$ $\frac{31}{8}$ $\frac{10}{1}$ $\frac{5}{2}$

* MANY POSSIBILITIES.

V



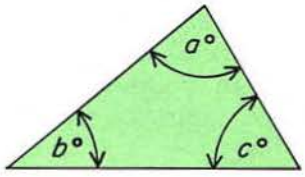
From the information given, indicate on the line above the cities from west to east.

Dodge City is west of Badger, but east of Malta. Justice is west of Benton, but east of Badger. Danville is between Malta and Dodge City.

Could you neatly tile a floor using triangular tiles all the same size and shape? We assume that you can cement pieces onto the floor with either side up. Also, you need not worry about the corners or the edges of the room.

Draw a triangular shape on the top sheet of several sheets of paper. Cut as many at a time as possible — so you will have a good supply. (It's more fun if no two angles of your triangular shape are the same size.)

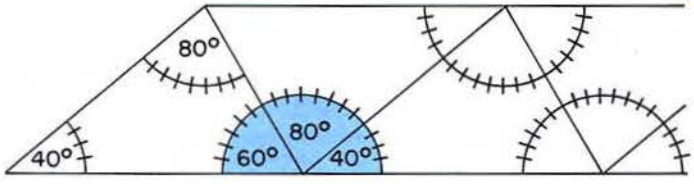
Here is one example:



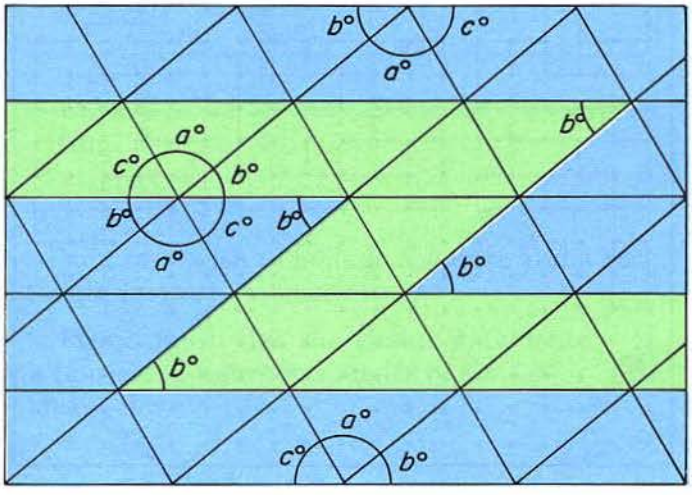
$a = 80$ $b = 40$ $c = 60$

Can you tell without using a protractor?

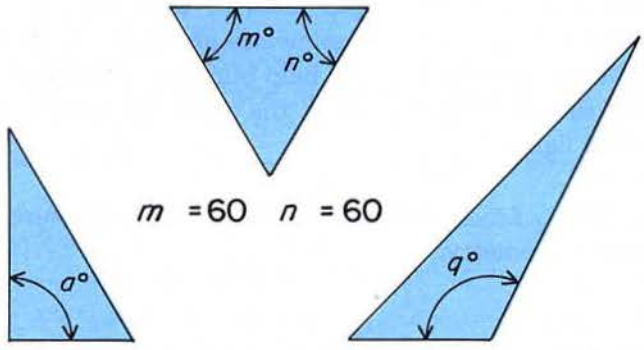
Here is a bridge design:



A pattern for a floor of tile the same shape (but smaller) will look like this, some green and some blue.



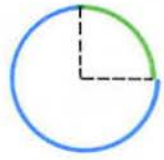
Make some floor covering designs of your own. Paste chunks of these designs on this page. Each time you finish a design, try a different triangular shape. Here are some you might like to choose from:



$m = 60$ $n = 60$

$a = 90$

q is more than 90

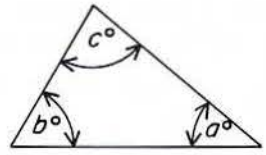


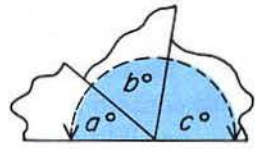
You know that the degree-measure of a quarter of a circle is 90.

If the degree-measure of a quarter of a circle is 90 then a half of a circle measures 2×90 or 180 degrees.

The experiments suggested in these pages have probably led you to the conclusion that the sum of the degree-measures of the angles of a triangle is equal to the degree-measure of a half of a circle.

This conclusion is a most useful fact to remember.





In shorthand, we write: If a triangle has angles of sizes $a°$, $b°$, and $c°$ then

$$a° + b° + c° = 180°$$

55 Experiments with triangular tiles (The sum of the degree measures of the angles of a triangle is 180.)

Tasks: *Pupils study a pattern of floor tiling and design a tile floor of their own with triangles of the same size and shape.*

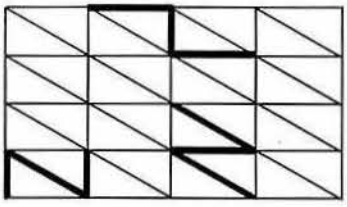
Purpose: To help pupils discover that any two triangles of the same size and shape can be arranged in such a way that they form a parallelogram, and that such triangles and parallelograms can be fitted to cover a floor. To help pupils conclude that the sum of the measures of the angles of any triangle is 180°.

Unifying Ideas: Functions and relations; measurement; geometry.

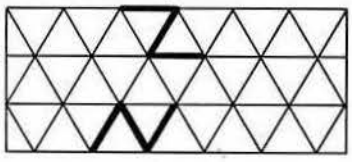
The Lesson: A floor can be covered with tiles of *any* triangular shape. Two such tiles can always be placed in such a way as to form a parallelogram. A series of parallelograms can then be fitted together to cover the floor. (Of course, the edges and corners might have to be trimmed to fit the rectangular shape of an actual floor. If you think of the floor as being of infinite extent—that is, as being a plane—the floor can indeed be tiled with triangles.)

Here are designs of such triangular tile patterns. Notice that in every such pattern, there are several examples of the "Z-result."

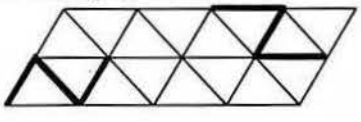
The right triangle:



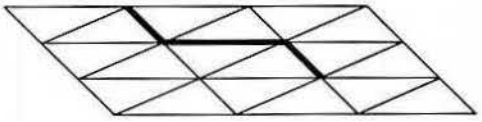
The equilateral triangle:



The acute triangle:



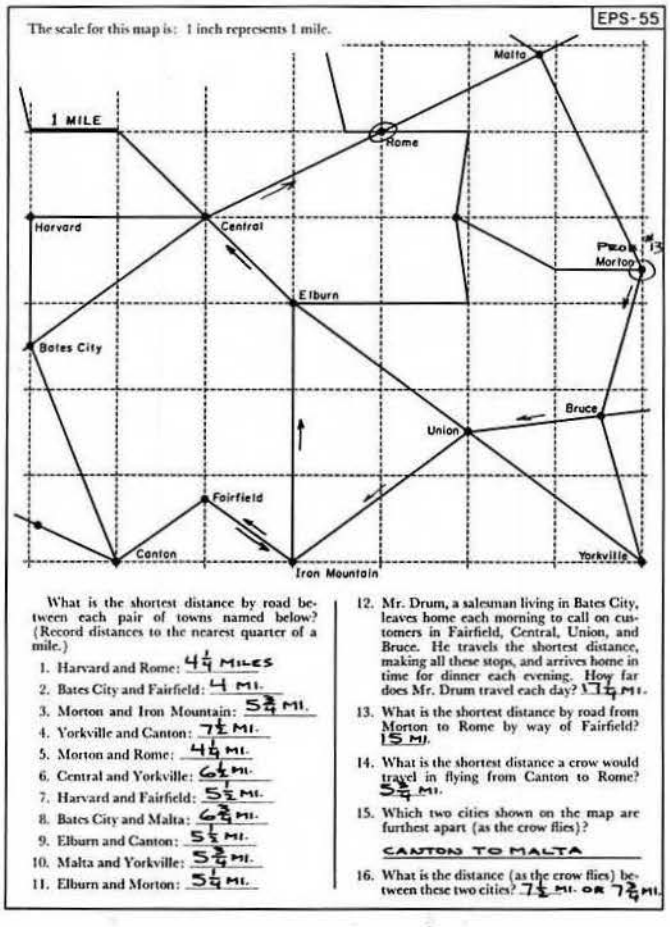
The obtuse triangle:



In the design at the bottom of the left-hand column on pupil page 55, the green color helps point out a "Z-result" with regard to the angles labeled $b°$.

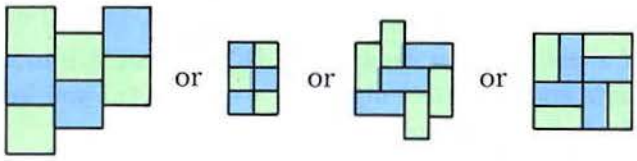
Here is a most difficult but interesting problem relating to practice sheet 55 that some of the more interested pupils may like to explore. Suppose that a salesman must work out the shortest route that will take him from Union to all other twelve towns and back to Union. What is that route and what is the total distance? Have pupils compare notes to see who has worked out the shortest route.

This leads to another question: Would there be a shorter route if the salesman moved his home to another city? Again, let the pupils compare results.



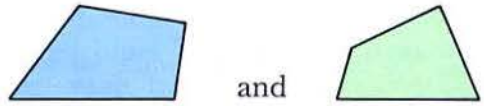
Experiments with Four-sided Tiles

Of course you can work out many tile patterns with square or rectangular tiles (four-sided tiles).



. . . and many others!

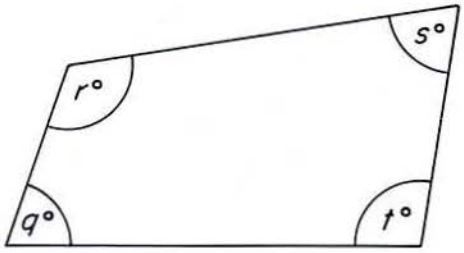
It's harder when no two sides of the four-sided tiles are equal in length. Try shapes like these:



Imagine that you have a large supply of tiles shaped like those shown above. Could you cover a floor with such tiles?

NOTE: Try your experiment before going on. If you need help, you may find some hints below.

In experiments we found that the sum of the degree-measures of the angles of a triangle is 180. In a four-sided figure, there are four angles. In the example below, what do you think $q^\circ + r^\circ + s^\circ + t^\circ$ will be?



Measure the angles of several four-sided figures. Keep a record of your measurements.

	The above example	Your experiments I	II	III	IV
q°					
r°					
s°					
t°					
Total					

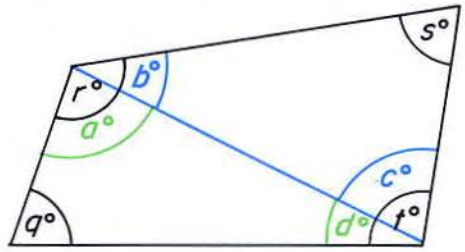
You may wish to include a square and a rectangle in your experiments. . . .

Please finish this shorthand statement: If a four-sided figure has angles of sizes q°, r°, s°, and t° then

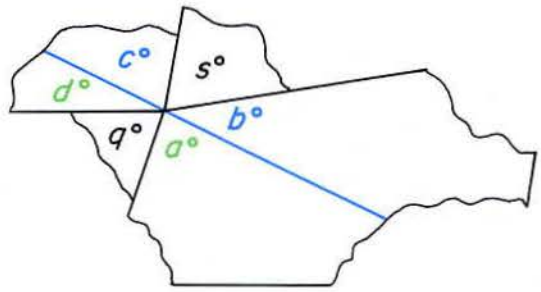
$q^\circ + r^\circ + s^\circ + t^\circ = 360^\circ$

Can you reach the same conclusion by reasoning, without any measuring?

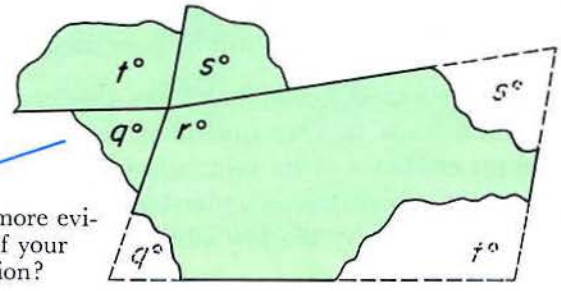
Draw a line that separates the above figure into two triangles. Does this give you any ideas?



If we were to tear off three corners and fitted them with the other, we might have:



The diagram below shows how you can tile the floor.



Is this more evidence of your conclusion?

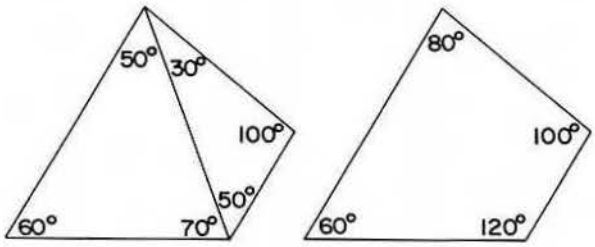
56 Experiments with four-sided tiles (The sum of the degree measures of the angles of a quadrilateral is 360.)

Tasks: *Pupils design a tile floor of their own with four-sided figures of the same size and shape and measure the angles of several four-sided figures.*

Purpose: To help pupils discover that four-sided (convex) figures of the same size and shape can be fitted to cover a floor. To help pupils conclude that the sum of the measures of the angles of any four-sided figure is 360°.

Unifying Ideas: Functions and relations; measurement; geometry.

The Lesson: The angle-measures of a four-sided (convex) figure total 360°.



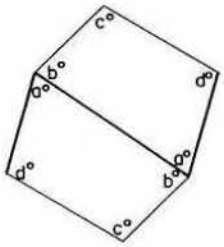
We can see why this is so if we draw a diagonal in the four-sided figure and thus form two triangles. In the example shown above:

$$180° = 60° + 50° + 70°$$
$$180° = 30° + 50° + 100°$$
$$\text{Add: } 360° = 60° + 80° + 120° + 100°$$

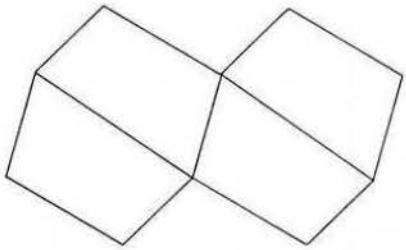
The 50° angle and the 30° angle joined together form an angle of 80°; the 70° angle and 50° angle joined together form an angle of 120°. The four angles in the right-hand figure above contain as many degrees as there are in the angles of the two triangles. One triangle's angles contain a total of 180°; the angles of the two triangles contain a total of 360°. Therefore, the angles of the four-sided figure also contain a total of 360°.

It is possible to make use of the above idea in tiling a floor. It may surprise pupils that this is possible. Have them experiment in the following way:

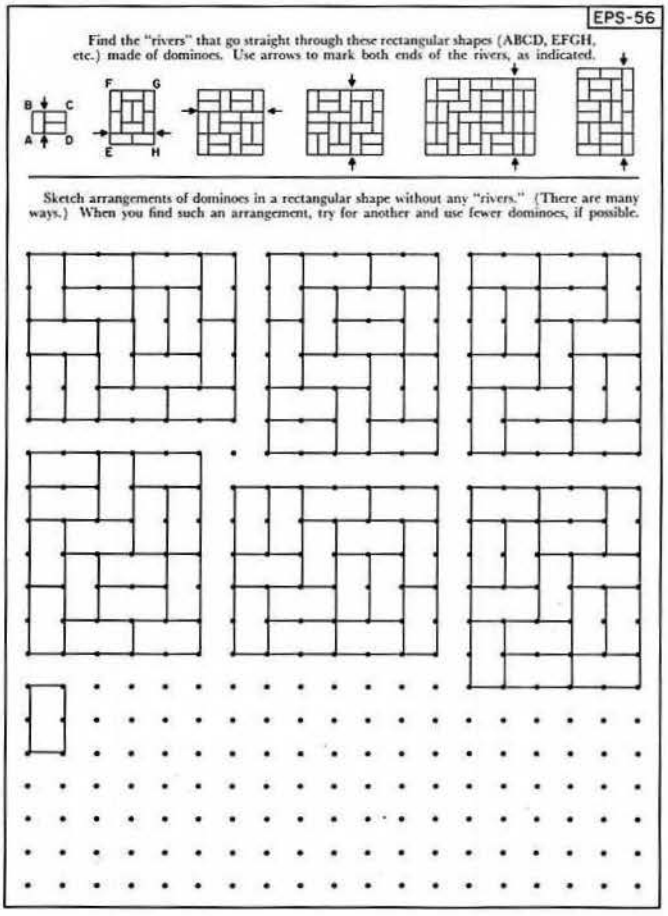
Begin with two tiles, cut to the same shape. Arrange the angles of $a°$ so that they are the angles of a "Z":



You now have a six-sided figure which has three pairs of parallel sides. Suppose that we call this figure a cell, because it is a little like the cell in a honeycomb. Make several of these cells from pairs of tiles. Place them side by side, like this:

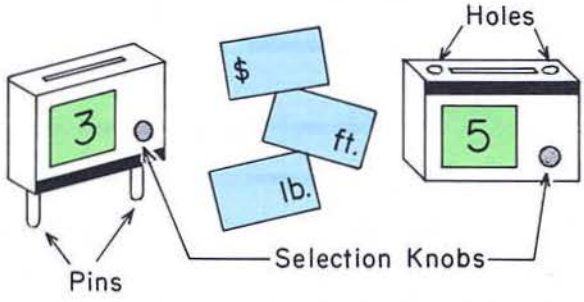


Make several rows in this manner. Next, push the rows together so that no spaces are left.



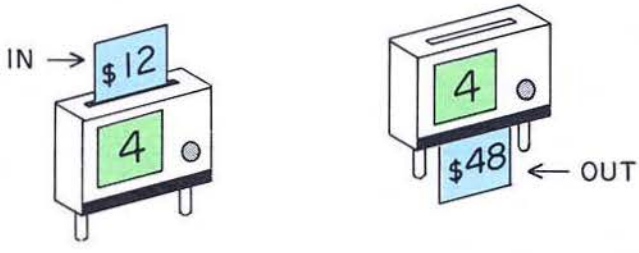
Alec's Magic Card Machine

Alec left a collection of small machines and a supply of magic cards. Here are sketches of two of his machines and some of his magic cards.



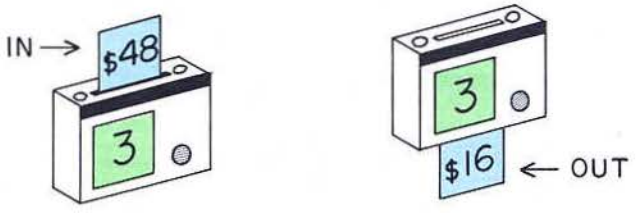
After some experimenting, Mr. Wilson found a way to use the machines. If he wrote an amount on one of the magic cards and put it in the slot on top, another magic card would come out at the bottom.

First he tried the machine with the black bar and the two pins at the bottom. He used the selection knob to show 4 in the green window. On a card with a dollar sign, he wrote 12 . . . and put it through the machine.



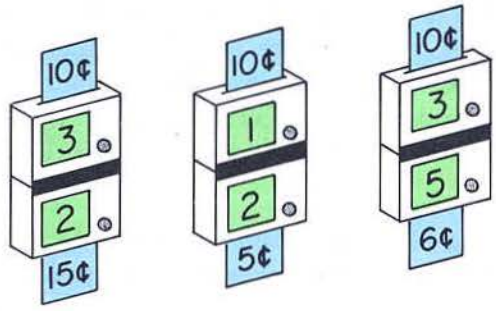
"A surprising machine," Mr. Wilson commented, "but not a surprising result."

Here is a similar experiment. This time the machine has two holes and the black bar at the top.



"A few more experiments led me to call one machine a 'Multiplier' and the other a 'Divider.'

"I noticed I could plug one machine into the other (the pins fit the holes). I wrote 10¢ on three cards and put them through the combined machines, turning the selector knobs between each trial. Here is a sketch of the results:



"I thought I knew just what the machines were doing. They were multiplying the 10¢ by the numbers I selected for the top machine and dividing the result by the numbers I selected for the bottom machine. I made notes:

IN		Top Setting		Same		Bottom Setting		OUT
10¢	x	3	=	30¢	and	30¢	÷ 2	= 15¢
10¢	x	1	=	10¢	and	10¢	÷ 2	= 5¢
10¢	x	3	=	30¢	and	30¢	÷ 5	= 6¢

"But I was less sure of just what the machines were doing when I reversed the order in which I used the machines. What would happen if I began by dividing 10¢ by the number I selected for the bottom machine? Let's try it.

IN			Same			OUT
10¢	÷ 2	=	5¢	and	5¢ x 3	= 15¢
10¢	÷ 2	=	5¢	and	5¢ x 1	= 5¢
10¢	÷ 5	=	2¢	and	2¢ x 3	= 6¢

"I found a page in Alec's notebook that helped me figure out how Alec used his machines."

57 Alec's Magic Card Machine (Multiplier machines, divider machines, and hookups of such machines)

Tasks: *Pupils read the story and fill the blanks in the right-hand column.*

Purpose: To lay the groundwork for an operator approach to fractions.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: Primary grade teachers work very hard to provide experiences from which pupils can begin to abstract the ideas of twoness, threeness, eightness, etc. When pupils grasp the comomn property of a pair of gloves, two fingers, a team of horses, they begin to understand the property of twoness.

Later, they learn to count groups of things and then combinations of groups. Whenever a group of two things and an entirely separate group of three things are joined and counted, the result is always the same. We mean to capture the common aspects of those counting experiments when we write $2+3=5$.

When a pupil is asked how many clowns there would be if two clowns were joined by three more clowns, and he puts up two fingers on one hand and three on the other, he has made a tremendous intellectual leap toward the great generality of the assertion that $2+3=5$. A question about clowns can be solved on the fingers!

If primary teachers err at all, it is in trying to rush the process of abstraction. They build a fundamental understanding when they help pupils realize that arithmetic is a means we have developed to mirror and report important aspects of experimenting with counting groups of things.

In this development of fractions, we propose to delay the important goal of abstraction until pupils have a strong foundation for that abstraction. Consider the following statements:

A half-dozen eggs is 6 eggs.
Half a dollar is 50 cents.
Half a half-gallon is a quart.
Bill can swim half as fast as his brother.
Halfway to the moon is more than 100,000 miles.
No halfway measures will be permitted.

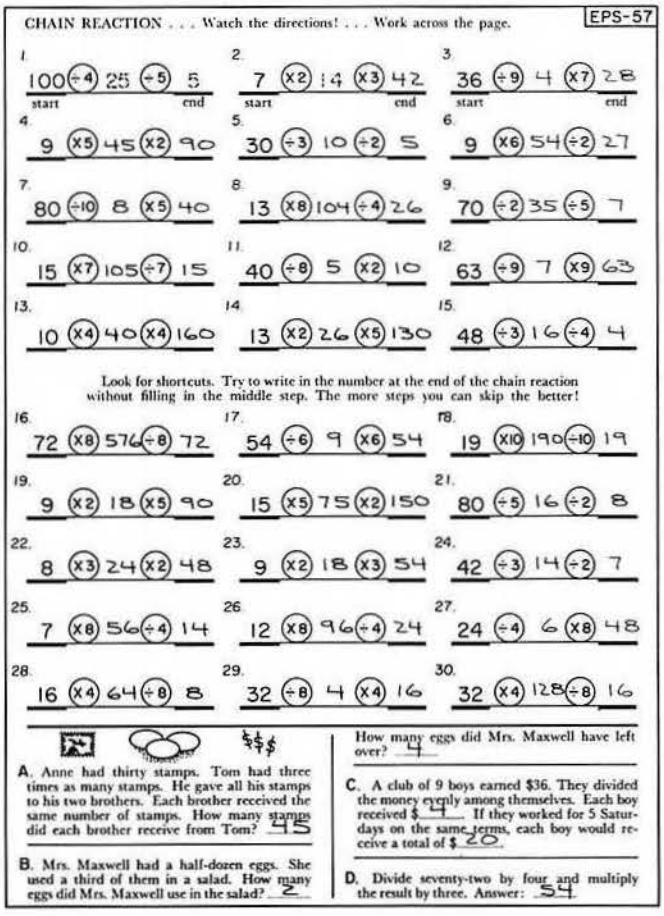
In each statement, the "half" has a different referent and leads to very dissimilar results. That ½ has an existence of its own, apart from any reference to something, is understood clearly by very few pupils and by too few adults.

We begin with the intuitive notion of fractions as instructions to do something to things or to measurements. If you wish to illustrate the idea of ¾ of a pie, (a) you draw a pie, (b) you cut it into 4 parts and indicate 1 of them ($\div 4$), and (c) you indicate 3 of those parts ($\times 3$).



Pupil page 57 and practice sheet 57 are designed to begin looking at this idea—at pairs of instructions (operations) to divide something by a number and to multiply the result by another number. In technical language, we are composing a *multiplication* function with a division function and applying the result to a quantity. This is what we shall mean by "taking a fraction" of the quantity.

We shall make no attempt to disguise the fact that we are considering the idea of fractions: the following page will dispel any such notion.



EPS-57

CHAIN REACTION . . . Watch the directions! . . . Work across the page.

1. 100 (÷4) 25 (÷5) 5 (start / end)	2. 7 (×2) 14 (×3) 42 (start / end)	3. 36 (÷9) 4 (×7) 28 (start / end)
4. 9 (×5) 45 (×2) 90	5. 30 (÷3) 10 (÷2) 5	6. 9 (×6) 54 (÷2) 27
7. 80 (÷10) 8 (×5) 40	8. 13 (×8) 104 (÷4) 26	9. 70 (÷2) 35 (÷5) 7
10. 15 (×7) 105 (÷7) 15	11. 40 (÷8) 5 (×2) 10	12. 63 (÷9) 7 (×9) 63
13. 10 (×4) 40 (×4) 160	14. 13 (×2) 26 (×5) 130	15. 48 (÷3) 16 (÷4) 4

Look for shortcuts. Try to write in the number at the end of the chain reaction without filling in the middle step. The more steps you can skip the better!

16. 72 (×8) 576 (÷8) 72	17. 54 (÷6) 9 (×6) 54	18. 19 (×10) 190 (÷10) 19
19. 9 (×2) 18 (×5) 90	20. 15 (×5) 75 (×2) 150	21. 80 (÷5) 16 (÷2) 8
22. 8 (×3) 24 (×2) 48	23. 9 (×2) 18 (×3) 54	24. 42 (÷3) 14 (÷2) 7
25. 7 (×8) 56 (÷4) 14	26. 12 (×8) 96 (÷4) 24	27. 24 (÷4) 6 (×8) 48
28. 16 (×4) 64 (÷8) 8	29. 32 (÷8) 4 (×4) 16	30. 32 (×4) 128 (÷8) 16

A. Anne had thirty stamps. Tom had three times as many stamps. He gave all his stamps to his two brothers. Each brother received the same number of stamps. How many stamps did each brother receive from Tom? 45

B. Mrs. Maxwell had a half-dozen eggs. She used a third of them in a salad. How many eggs did Mrs. Maxwell use in the salad? 2

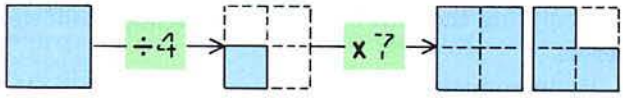
How many eggs did Mrs. Maxwell have left over? 4

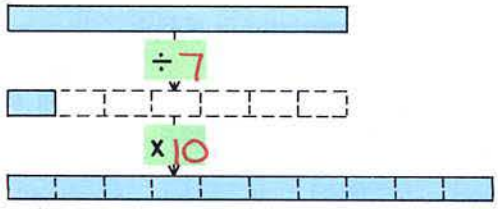
C. A club of 9 boys earned $36. They divided the money evenly among themselves. Each boy received $ 4. If they worked for 5 Saturdays on the same terms, each boy would receive a total of $ 20.

D. Divide seventy-two by four and multiply the result by three. Answer: 54

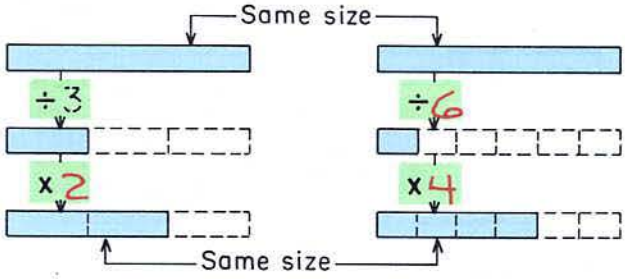
When I see a fraction — $\frac{3}{5}$, for example — I think of it as telling me to divide something by 5 and multiply the result by 3. I think of some sketches which will help me to understand the idea better. *(Please fill in any missing instructions.)*

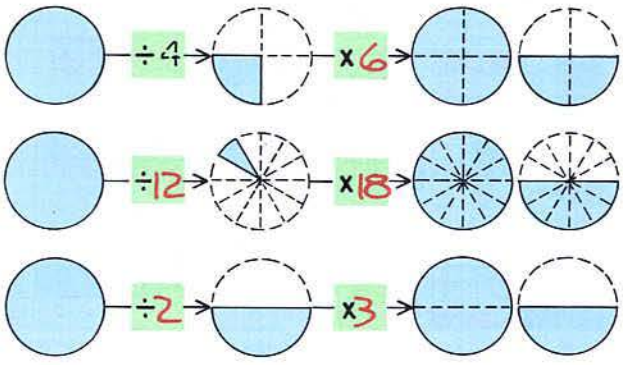






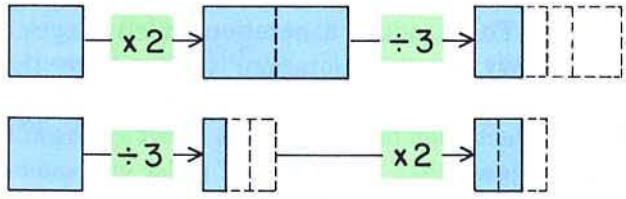
Special kinds of sketches illustrate the idea that different pairs of instructions may lead to the same final result.

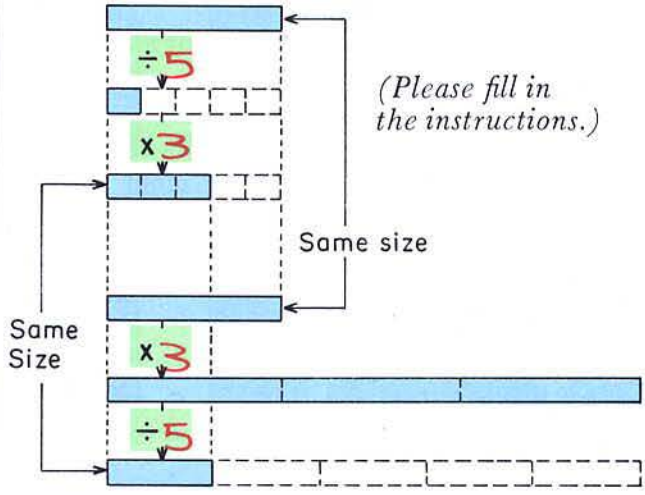




(You may wish to draw sketches of your own to show that different pairs of instructions may lead to the same final result.)

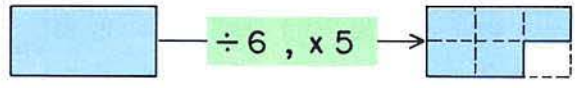
Sketches can also help to convince me that the order of carrying out a pair of instructions does not affect the final result.



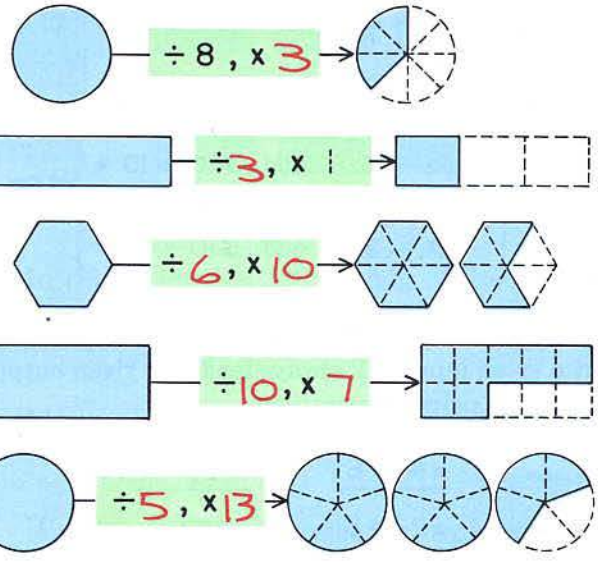


(Please fill in the instructions.)

I can simplify my sketches by doing the following:



(Please fill in missing instructions.)



58 A page from Alec's notebook (Comparing Alec's machine operations to taking fractions of areas of pies, rectangular strips, and hexagonal regions)

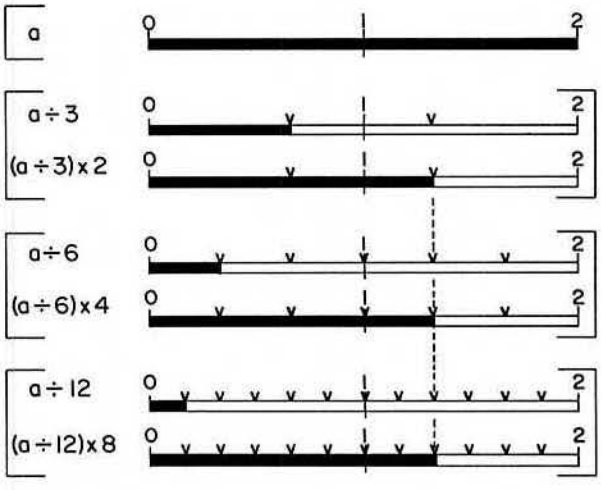
Tasks: *Pupils specify the instructions (division and multiplication operations) by writing numbers next to the given signs.*

Purpose: To lay the groundwork for an operator approach to fractions; to illustrate the equivalence of certain pairs of operations; to illustrate the fact that the order of application of the operations in a pair does not affect the result (in other words, to illustrate that composition of multiplication and division operations is commutative); to introduce a shorthand notation for specifying a pair of operations.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

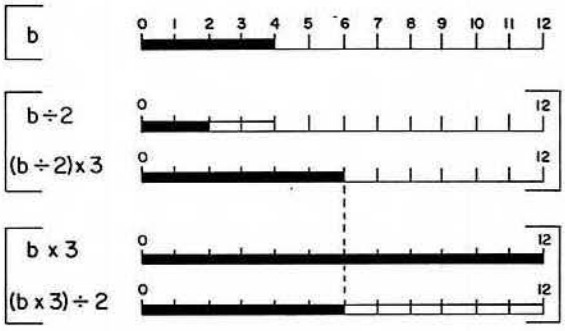
The Lesson: Alec, in his notebook, explains his understanding of a fraction as a pair of instructions to divide something by a number and then to multiply the result by another number. In graphic form, he notices two fundamental results of his experiments: (1) that, for any given region, many different pairs of operations lead to the same result; and (2) that the order in which the operations are carried out does not affect the result.

You may wish to explore the first idea on a number line. Suppose that we start with the distance between 0 and 2:



Practice sheet 58 used a similar method of graphic representation.

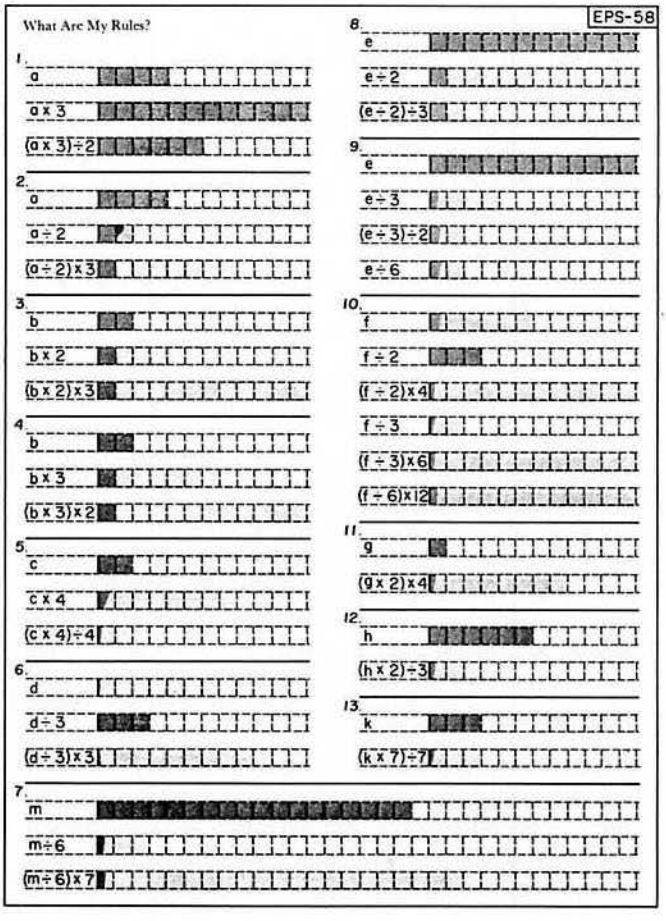
The number line can also be used to explore the second idea—that the order of carrying out the multiplication and division operations does not affect the result.



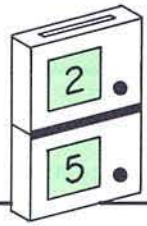
Neither of these ideas is obvious. Time spent on the pupil page, on practice sheets 58 and 59, and on many experiments with the number line will be well invested, since these two ideas are essential to understanding of two related ideas about fractions.

1. That there are many equivalent pairs of operations underlies the notion of equivalent fractions: $\frac{2}{3} = \frac{4}{6} = \frac{8}{12}$.
2. That the order in carrying out a pair of operations does not affect the result is the basis for shortcuts in carrying out computations with fractions:

$$\cancel{9}^{3} \times \frac{2}{\cancel{3}} = 6 \quad \text{instead of:} \quad 9 \times \frac{2}{3} = \frac{18}{3} = 6$$



Mr. Wilson experiments with Alec's "Fraction Machine"



"As I began working with Alec's machine," Mr. Wilson told the class, "I developed a shorthand to keep track of my results.

"In the beginning, I used a little multiplication sign next to my record of the top machine-setting and a little division sign next to my record of the bottom machine-setting.

"Here are a few examples."

Please supply the missing entries.

IN →	14¢	8 ft.	$10	15 oz.
	x 1	x 3	x 9	x 7
	÷ 2	÷ 4	÷ 5	÷ 3
OUT→	7¢	6 ft.	$18	35 oz.

"I soon learned that frequently the arithmetic was easier if I carried out the division first and then multiplied.

"Alec's notes suggested that several settings of the selectors could produce the same final results. My experiments showed he was right."

30¢	30¢	30¢	30¢
x 4	x 10	x 2	x 6
÷ 6	÷ 15	÷ 3	÷ 9
20¢	20¢	20¢	20¢

"That last example reminded me that sometimes it is more convenient to multiply first and then divide. $30¢ \div 9$ is not a whole number of cents; but $180¢ \div 9 = 20¢$."

Mr. Wilson gave the class some practice in deciding which pairs of machine-settings produced the same results. He gave an input card and an output card. Then he wrote out several pairs of machine-settings. He explained:

"Check (✓) machine-settings that would produce the results shown; cross out (×) those that would not. Then write the machine-setting with the smallest numbers in the boxes on the left."

12 oz.	x 2 / ÷ 3	x 2 / ÷ 4	x 1 / ÷ 2	x 3 / ÷ 4
x 1	x 6 / ÷ 1	x 3 / ÷ 6	x 12 / ÷ 6	x 5 / ÷ 10
÷ 2				
6 oz.				

24 in.	x 3 / ÷ 2	x 2 / ÷ 3	x 12 / ÷ 8	x 9 / ÷ 6
x 3	x 21 / ÷ 14	x 6 / ÷ 4	x 3 / ÷ 12	x 15 / ÷ 10
÷ 2				
36 in.				

$60	x 18 / ÷ 30	x 6 / ÷ 10	x 10 / ÷ 15	x 15 / ÷ 25
x 6	x 9 / ÷ 15	x 16 / ÷ 3	x 20 / ÷ 12	x 12 / ÷ 20
÷ 10				
$36				

72¢	x 18 / ÷ 28	x 7 / ÷ 4	x 54 / ÷ 35	x 42 / ÷ 24
x 7	x 14 / ÷ 8	x 63 / ÷ 36	x 21 / ÷ 12	x 8 / ÷ 14
÷ 4				
126¢				

In the pages that follow, Mr. Wilson carries out many experiments with Alec's "Fraction Machine."

Many of his entries are missing. Please fill them in.

Perhaps you might make up experiments of your own.

59 Experimenting with Alec's fraction machine (Exploring equivalent fractions)

Tasks: *Pupils compute missing entries and select equivalent fractions.*

Purpose: To introduce a notation which suggests the ordinary fraction notation; to illustrate the equivalence of certain pairs of operations; to illustrate the fact that order does not affect the result and to suggest that the choice of order can sometimes lead to a shortcut; to lay the groundwork for reducing a fraction to lowest terms.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: The idea that several pairs of instructions (operations) can link a given input with a given output is explained on pupil page 59, using examples from money and weight systems.

Several pairs of multiplication and division operations lead from 30¢ to 20¢.

$$30¢ \left\{ \begin{array}{llll} \times 4 = 120¢ & \text{and} & 120¢ \div 6 = \\ \times 10 = 300¢ & \text{and} & 300¢ \div 15 = \\ \times 2 = 60¢ & \text{and} & 60¢ \div 3 = \\ \times 6 = 180¢ & \text{and} & 180¢ \div 9 = \end{array} \right\} 20¢$$

Try these same pairs with a different input.

$$12\text{ in.} \left\{ \begin{array}{llll} \times 4 = 48\text{ in.} & \text{and} & 48\text{ in.} \div 6 = \\ \times 10 = 120\text{ in.} & \text{and} & 120\text{ in.} \div 15 = \\ \times 2 = 24\text{ in.} & \text{and} & 24\text{ in.} \div 3 = \\ \times 6 = 72\text{ in.} & \text{and} & 72\text{ in.} \div 9 = \end{array} \right\} 8\text{ in.}$$

Try reversing the order and selecting still another input.

$$\$90 \left\{ \begin{array}{llll} \div 6 = \$15 & \text{and} & \$15 \times 4 = \\ \div 15 = \$6 & \text{and} & \$6 \times 10 = \\ \div 3 = \$30 & \text{and} & \$30 \times 2 = \\ \div 9 = \$10 & \text{and} & \$10 \times 6 = \end{array} \right\} \$60$$

The inputs may change, and the order may change, but a given input will always lead to a given output. Thus, four pairs of operations

$\underline{\times 4, \div 6}$ $\quad$ $\underline{\times 10, \div 15}$

$\underline{\times 2, \div 3}$ $\quad$ $\underline{\times 6, \div 9}$

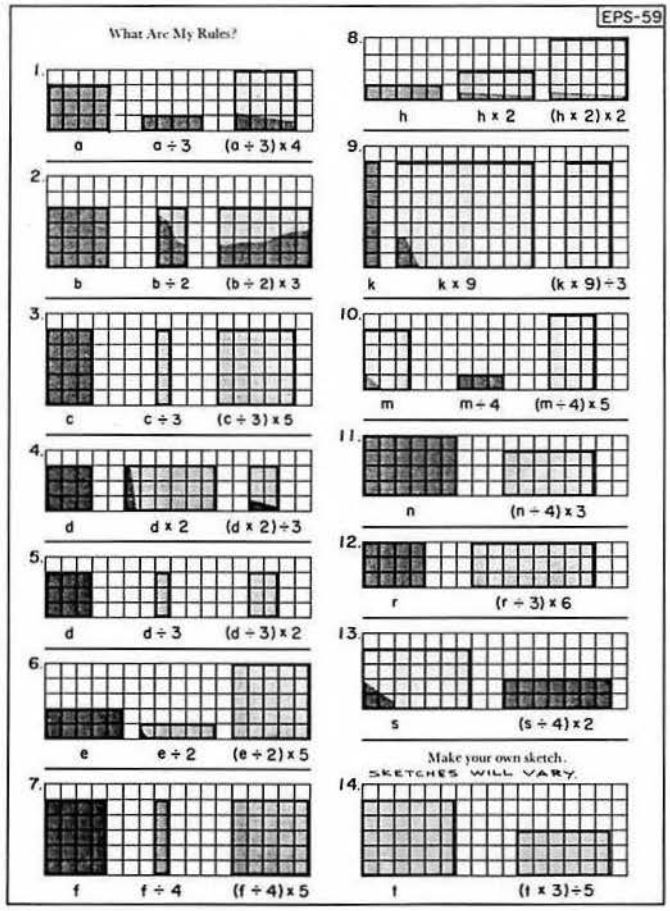
have something in common—and begin to demonstrate some independence of the changing inputs. Later on, pupils will have a deeper appreciation of what we mean when we write:

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{10}{15} \cdots \text{etc.}$$

In the right-hand column, pupils select the simplest pair of instructions in each of the four examples, anticipating the idea of reducing fractions to lowest terms.

By introducing fractions as pairs of multiplication and division operations, we ask pupils to do nothing more than multiply and divide by the familiar whole numbers. In this way, pupils simply follow a pair of familiar instructions.

All operations with fractions—including division of one fraction by another—can be understood in terms of pairs of operations with whole numbers. We shall continue to base our arguments on simple, familiar, and easily verifiable ideas.



Another Investigation

"Often when I was experimenting with Alec's machines, I would use the same input with many different settings on the multiplier and divider machines (in green)."



12 EGGS

x 1 ÷ 2	x 1 ÷ 3	x 2 ÷ 3	x 3 ÷ 3	x 1 ÷ 4	x 2 ÷ 4	x 3 ÷ 4	x 4 ÷ 4	x 1 ÷ 6	x 2 ÷ 6	x 3 ÷ 6
6	4	8	12	3	6	9	12	2	4	6

All numbers written in blue blocks are numbers of eggs.

x 4 ÷ 6	x 5 ÷ 6	x 6 ÷ 6	x 1 ÷ 12	x 2 ÷ 12	x 3 ÷ 12	x 4 ÷ 12	x 5 ÷ 12	x 6 ÷ 12	x 7 ÷ 12	x 8 ÷ 12	x 9 ÷ 12	x 10 ÷ 12	x 11 ÷ 12
8	10	12	1	2	3	4	5	6	7	8	9	10	11

x 12 ÷ 12	x 2 ÷ 2	x 3 ÷ 2	x 4 ÷ 3	x 5 ÷ 4	x 6 ÷ 4	x 4 ÷ 2	x 8 ÷ 6	x 6 ÷ 2	x 9 ÷ 6	x 8 ÷ 4	x 7 ÷ 3	x 15 ÷ 12	x 5 ÷ 2
12	12	18	16	15	18	24	16	36	18	24	28	15	30

"Here are some machine-settings that gave me results."

6	x 1 ÷ 2	x 2 ÷ 4	x 3 ÷ 6	x 6 ÷ 12

4	x 1 ÷ 3	x 2 ÷ 6	x 4 ÷ 12

8	x 2 ÷ 3	x 4 ÷ 6	x 8 ÷ 12

3	x 1 ÷ 4	x 3 ÷ 12

9	x 3 ÷ 4	x 9 ÷ 12

2	x 1 ÷ 6	x 2 ÷ 12

10	x 5 ÷ 6	x 10 ÷ 12

12	x 3 ÷ 3	x 4 ÷ 4	x 6 ÷ 6	x 12 ÷ 12	x 2 ÷ 2

18	x 3 ÷ 2	x 6 ÷ 4	x 9 ÷ 6

24	x 4 ÷ 2	x 8 ÷ 4



10 CENTS

x 1 ÷ 10	x 1 ÷ 5	x 1 ÷ 2	x 2 ÷ 10	x 2 ÷ 5	x 2 ÷ 2	x 3 ÷ 10	x 3 ÷ 5	x 3 ÷ 2	x 4 ÷ 10	x 4 ÷ 5
1	2	5	2	4	10	3	6	15	4	8

All numbers written in blue blocks are numbers of cents.

x 4 ÷ 2	x 5 ÷ 10	x 5 ÷ 5	x 10 ÷ 20	x 6 ÷ 10	x 15 ÷ 10	x 1 ÷ 1	x 7 ÷ 10	x 6 ÷ 15	x 2 ÷ 2	x 9 ÷ 15	x 5 ÷ 5	x 12 ÷ 20	x 10 ÷ 10
20	5	10	5	6	15	10	7	4	10	6	10	6	10

5	x 1 ÷ 2	x 5 ÷ 10	x 10 ÷ 20

4	x 2 ÷ 5	x 4 ÷ 10	x 6 ÷ 15

6	x 3 ÷ 5	x 6 ÷ 10	x 9 ÷ 15	x 12 ÷ 20

60 Another investigation into equivalent fractions, using a dozen eggs and a dime

Tasks: *Pupils compute missing entries and select equivalent fractions.*

Purpose: To practice finding fractions of given quantities; to practice selecting equivalent fractions from a given list of fractions.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: We often use simple fractions in everyday life. Such expressions as:

three-quarters of an hour
a quarter
seven-eighths of an inch
half a pound
a dollar and a half

are part of our casual conversation. We do not have to think twice to know their meaning. Less often, we encounter the following:

three-fourths of five pounds
three-quarters of two yards
three-quarters of $2.60
three-fourths of 72 apples

And now, even adults need a way to think about what we mean by three-quarters of some measurement or some count of things.

Whether you are explaining ¾ of an hour or ¾ of $2.60, you will undoubtedly present arguments such as the following:

One-quarter of 60 minutes is 15 minutes (÷4); so, three-quarters would be 45 minutes (×3). Divide $2.60 by 4 to find one-quarter; this is 65¢; multiply that by 3 . . . 65¢ × 3 = $1.95.

"Divide by 4 and multiply by 3" is the way we think about ¾, whenever we need to think about it at all.

Pupil page 60 explores in depth what happens when we apply pairs of operations to a dozen eggs and to the value of a dime.

A dozen eggs can be partitioned into groups with the same number of eggs per group in several ways—groups of 12, of 6, of 4, of 3, of 2, and of 1. Such partitions are completely familiar to the pupils. Further, they understand what we mean by, say, 7 groups with 4 eggs each—or 28 eggs. So, pupil page 60 requires nothing more than keeping records of experiences pupils already understand fully.

However, attention is focused on an interesting outcome: several pairs of operations when applied to a dozen eggs lead to the same outcome. Each of the pairs

×2, ÷3 ×4, ÷6 ×8, ÷12

takes us from 12 eggs to 8 eggs. This fact is to be recorded on the pupil page in:

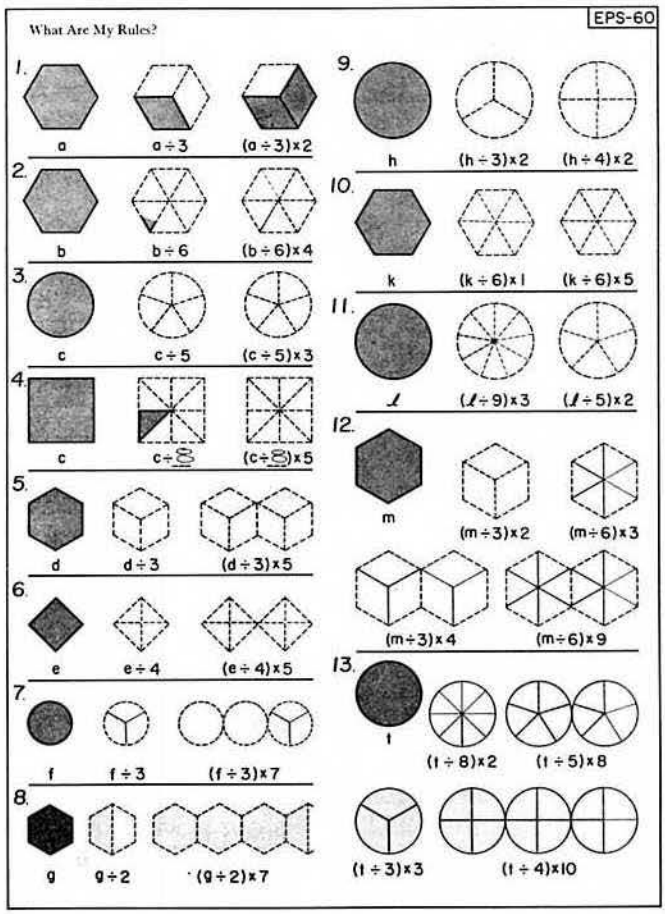
8	×2 ÷	× ÷	× ÷

Notice a twist when we apply the operations

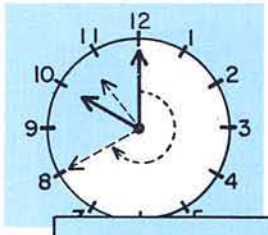
×9, ÷15

to 10¢. We may have been dividing first and then multiplying—because that's easier. In this case, if we wish to avoid considering a part of a cent, we must multiply first, 10¢ × 9 = 90¢, and then divide, 90¢ ÷ 15 = 6¢.

On practice sheet 60 pupils continue to apply pairs of operations to areas of regions of various shapes and sizes.



"Since Alec always put the multiplier machines on top and divider machines on the bottom, I began to leave out the ' × ' signs and the ' ÷ ' signs."



40 minutes

$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{7}{5}$
20	40	60	10	20	30	40	50	60	8	16	24	32	56

Numbers written in blue blocks are numbers of minutes.

$\frac{1}{8}$	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{6}{8}$	$\frac{8}{8}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{5}{10}$	$\frac{5}{20}$	$\frac{10}{20}$	$\frac{30}{20}$	$\frac{2}{20}$	$\frac{12}{8}$	$\frac{15}{10}$	$\frac{1}{40}$	$\frac{4}{40}$	$\frac{50}{40}$	$\frac{60}{40}$
5	10	20	30	40	4	8	20	10	20	60	4	60	60	1	4	50	60

Groups of machine-settings that give the same results.

20	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{4}{8}$	$\frac{5}{10}$	$\frac{10}{20}$

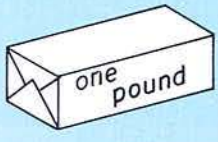
30	$\frac{3}{4}$	$\frac{6}{8}$

4	$\frac{1}{10}$	$\frac{2}{20}$	$\frac{4}{40}$

10	$\frac{1}{4}$	$\frac{2}{8}$	$\frac{5}{20}$

40	$\frac{2}{2}$	$\frac{4}{4}$	$\frac{8}{8}$

60	$\frac{3}{2}$	$\frac{6}{4}$	$\frac{30}{20}$	$\frac{12}{8}$	$\frac{15}{10}$	$\frac{60}{40}$



16 ounces

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{2}{1}$	$\frac{5}{2}$	$\frac{6}{4}$	$\frac{2}{8}$	$\frac{10}{4}$	$\frac{3}{1}$	$\frac{2}{4}$	$\frac{20}{8}$	$\frac{6}{8}$	$\frac{4}{2}$
8	4	12	24	32	40	24	4	40	48	8	40	12	32

Numbers written in blue blocks are numbers of ounces.

$\frac{10}{2}$	$\frac{8}{4}$	$\frac{4}{16}$	$\frac{12}{4}$	$\frac{24}{16}$	$\frac{7}{4}$	$\frac{4}{8}$	$\frac{40}{16}$	$\frac{16}{8}$	$\frac{12}{8}$	$\frac{40}{8}$	$\frac{8}{16}$	$\frac{24}{8}$	$\frac{32}{16}$	$\frac{14}{8}$	$\frac{12}{16}$	$\frac{6}{2}$	$\frac{28}{16}$
80	32	4	48	24	28	8	40	32	24	80	8	48	32	28	12	48	28

Groups of machine-settings that give the same results.

8	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{4}{8}$	$\frac{8}{16}$

4	$\frac{1}{4}$	$\frac{2}{8}$	$\frac{4}{16}$

24	$\frac{3}{2}$	$\frac{6}{4}$	$\frac{24}{16}$	$\frac{12}{8}$

12	$\frac{3}{4}$	$\frac{6}{8}$	$\frac{12}{16}$

32	$\frac{2}{1}$	$\frac{4}{2}$	$\frac{8}{4}$	$\frac{16}{8}$	$\frac{32}{16}$

40	$\frac{5}{2}$	$\frac{10}{4}$	$\frac{20}{8}$	$\frac{40}{16}$

48	$\frac{3}{1}$	$\frac{12}{4}$	$\frac{24}{8}$	$\frac{6}{2}$

28	$\frac{7}{4}$	$\frac{14}{8}$	$\frac{28}{16}$

80	$\frac{10}{2}$	$\frac{40}{8}$	$\frac{80}{16}$

61 Continuing the investigation, using 40 minutes and one pound

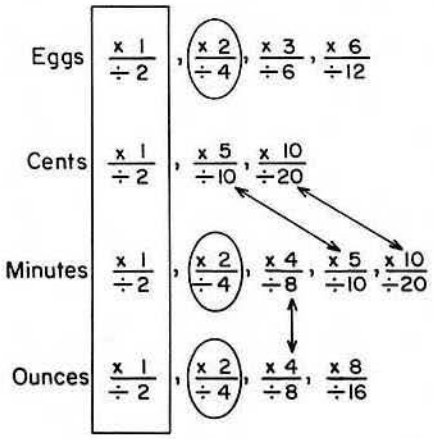
Tasks: *Pupils compute missing entries and select equivalent fractions.*

Purpose: To introduce fraction notation; to practice finding fractions of given quantities; to practice selecting equivalent fractions from a given list of fractions.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: Pairs of operations are applied to 40 minutes and to 16 ounces (a pound)—an extension of the activity on the previous page.

In each of the four investigations on pupil pages 60 and 61, there is a group of equivalent pairs of operations which includes the pair $\times 1$, $\div 2$.



The pair of operations

$$\times 2, \div 4$$

is included in three groups, and the pairs

$$\times 4, \div 8 \qquad \times 5, \div 10 \qquad \times 10, \div 20$$

are included in two groups each. Are any of these pairs of operations included in any other lists? Inspection reveals none.

Let's list all the pairs that are included in any of the four groups in order from the smallest number used in any pair to the largest:

$$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \frac{7}{14}, \frac{8}{16}, \frac{9}{18}, \frac{10}{20}$$

As we complete our list, we are tempted to leave blanks as above, and the pattern is strong enough to persuade us to fill in the gaps.

Consider making a similar list for the examples with eggs, minutes, and ounces:

$$\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \frac{5}{20}, \_\_, \_\_, \_\_, \_\_$$

How would you extend this list? Which would you call the simplest of the pairs of operations?

On practice sheet 61 pupils make lists of multiples of numbers and look for numbers appearing in each of two lists, in each of three lists, and in each of four lists. Such numbers are called *common multiples*. For example, 72 is a common multiple of 4 and 9 because it appears in a list of multiples of 4 and in a list of multiples of 9. The smallest number in a list of common multiples of two or more given numbers is called the lowest, or least, common multiple of the given numbers. Lowest common multiples will reappear in our work as lowest, or least, common denominators.

What four numbers less than 10 have the largest least common multiple? One might be inclined to guess 6, 7, 8, and 9 since these are the four largest whole numbers less than 10. But this guess is wrong.

LCM of 6, 7, 8, and 9 is 504
LCM of 5, 7, 8, and 9 is 2,520

EPS-61

Record in the spaces provided multiples of the numbers listed. Arrange the multiples in increasing order so that the smallest multiple greater than zero is listed in the first space on the left.

multiples of 1 . . .	1	2	3	4	5	6	7	8	9	10
multiples of 2 . . .	2	4	6	8	10	12	14	16	18	20
multiples of 3 . . .	3	6	9	12	15	18	21	24	27	30
multiples of 4 . . .	4	8	12	16	20	24	28	32	36	40
multiples of 5 . . .	5	10	15	20	25	30	35	40	45	50
multiples of 6 . . .	6	12	18	24	30	36	42	48	54	60
multiples of 7 . . .	7	14	21	28	35	42	49	56	63	70
multiples of 8 . . .	8	16	24	32	40	48	56	64	72	80
multiples of 9 . . .	9	18	27	36	45	54	63	72	81	90
multiples of 10 . .	10	20	30	40	50	60	70	80	90	100

List numbers that are multiples (greater than 0) of both numbers listed on each line. List them in increasing order so that the smallest multiple appears in the first space on the left.

Multiples of

2 and 3 . . .	6	12	18	24
5 and 2 . . .	10	20	30	40
3 and 4 . . .	12	24	36	48
4 and 2 . . .	4	8	12	16
6 and 4 . . .	12	24	36	48
3 and 8 . . .	24	48	72	96
9 and 3 . . .	9	18	27	36
4 and 10 . .	20	40	60	80
6 and 9 . . .	18	36	54	72
7 and 2 . . .	14	28	42	56
7 and 14 . .	14	28	42	56
4 and 9 . . .	36	72	108	144

⇧

The numbers in this column are called the LOWEST COMMON MULTIPLE of the numbers given: 12 is the lowest common multiple of 6 and 4; etc.

List in increasing order the common multiples of the numbers indicated. List the lowest common multiple greater than zero in the first space on the left.

Multiples of

2, 3, and 6 . . .	6	12	18	24
2, 3, and 9 . . .	18	36	54	72
2, 4, and 5 . . .	20	40	60	80
3, 4, and 8 . . .	24	48	72	96
2, 3, and 5 . . .	30	60	90	120
2, 4, and 7 . . .	28	56	84	112
3, 5, and 9 . . .	45	90	135	180
4, 6, and 8 . . .	24	48	72	96
2, 5, and 7 . . .	70	140	210	280

Multiples of

2, 3, 4, and 5 . .	60	120	180	240
2, 4, 6, and 8 . .	24	48	72	96
3, 6, 9, and 12 .	36	72	108	144
2, 5, 10, and 15.	30	60	90	120
2, 3, 6, and 7 . .	42	84	126	168
2, 3, 5, and 7 . .	210	420	630	840

⇧

Lowest common multiple of numbers given.

"In the following records," Mr. Wilson explained, "I find several pairs of machine-settings that would produce the same result.

"I make sure that one pair of the machine-settings uses the smallest whole numbers possible. I mark this pair of machine-settings with a check (✓)."

Please complete these records.



✓			
2/1	4/2	8/4	10/5
56	56	56	56

✓		
3/4	21/28	6/8
21	21	21

✓		
7/14	1/2	14/28
14	14	14

✓	
1/4	7/28
7	7

Numbers written in blue blocks are numbers of days.

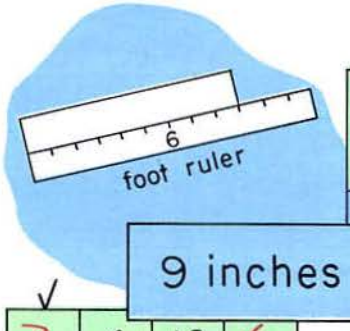
✓		
4/14	2/7	8/28
8	8	8

✓		
12/28	6/14	3/7
12	12	12

✓	+	
20/28	5/7	10/14
20	20	20

✓		+
1/1	2/2	4/4
28	28	28

✓	+	+	+
3/1	6/2	12/4	9/3
84	84	84	84



✓		
1/3	3/9	6/18
3	3	3

✓		
2/3	6/9	12/18
6	6	6

✓		
4/9	8/18	16/36
4	4	4

✓		
4/3	12/9	8/6
12	12	12

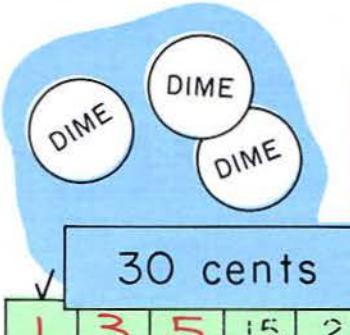
Numbers written in blue blocks are numbers of inches.

✓			
2/1	4/2	10/5	6/3
18	18	18	18

✓	+	+	+
3/1	6/2	9/3	12/4
27	27	27	27

✓	+	+	+
5/3	10/6	15/9	20/12
15	15	15	15

✓	+	+	+
5/9	10/18	15/27	20/36
5	5	5	5



Numbers written in blue blocks are numbers of cents.

✓						
1/3	2/6	10/30	5/15	4/12	3/9	6/18
10	10	10	10	10	10	10

✓						
2/5	4/10	12/30	6/15	8/20	10/25	
12	12	12	12	12	12	

✓					+	
1/2	3/6	5/10	15/30	2/4	4/8	8/16
15	15	15	15	15	15	15

✓				+	
2/3	4/6	8/12	12/18	6/9	10/15
20	20	20	20	20	20

✓			
1/1	30/30	2/2	6/6
30	30	30	30

✓	+	+	+	
1/10	2/20	3/30	4/40	5/50
3	3	3	3	3

✓	+		+
3/2	6/4	45/30	9/6
45	45	45	45

✓	+			+
2/1	4/2	6/3	8/4	10/5
60	60	60	60	60

✓	
11/30	22/60
11	11

+ MANY POSSIBLE ANSWERS.

Tasks: *Pupils compute equivalent fractions and search for fractions in lowest terms.*

Purpose: To practice computing equivalent fractions and reducing fractions to lowest terms.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: On pupil page 61 we dropped all the "×" and "÷" signs, by adopting the agreement that the number written above the line tells what to multiply by and the one below tells what to divide by. Thus, we did not drop the idea; rather, we improved our shorthand.

On pupil pages 62 and 63 pupils make lists of equivalent operations and then find the simplest pair—the pair consisting of the smallest whole numbers. These pairs are then recorded in the checked boxes.

Notice that in many instances neither member of the pair of operations is given. There are an unlimited number of correct entries in such cases. (We hope you will encourage a pupil who demonstrates his mastery of the idea by including the most imposing looking pair of operations.)

$$\frac{4}{9}, \frac{8}{18}, \frac{68}{153}, \frac{44{,}444}{99{,}999}, \text{ etc.}$$

To help the pupils who are less sure to see more deeply into these lists, you might start a list and leave gaps:

$$\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \ldots\ldots, \frac{?}{100}, \ldots\ldots, \frac{123}{?}$$

and rewrite the items:

$$\frac{3\times1}{5\times1}, \frac{3\times2}{5\times2}, \frac{3\times3}{5\times3}, \frac{3\times4}{5\times4}, \ldots\ldots, \frac{3\times20}{5\times20}, \ldots\ldots, \frac{3\times61}{5\times61}$$

$$\ldots\ldots, \frac{60}{100}, \ldots\ldots, \frac{123}{205}$$

Investigation of the internal pattern in such lists is extended on practice sheet 62. Enough is shown of the sixteen lists at the top of the page so that pupils could extend them if they felt the need. Then chunks out of these lists are shown, with many entries missing. Some of the situations require considerable appreciation for the internal relationships in the lists.

For example,

$$\frac{18}{48} = \frac{3\times6}{8\times6} \text{ and } \frac{3\times7}{8\times7}$$

and so the next entry must be

$$\frac{3\times7}{8\times7} = \frac{21}{56}$$

In the example:

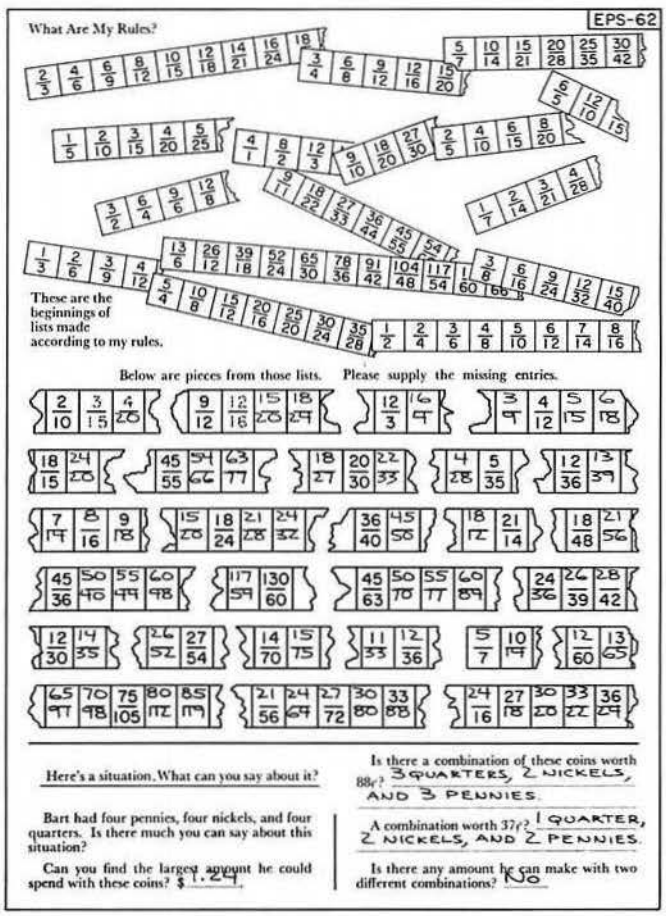
$$\frac{24}{\ }, \frac{\ }{39}, \frac{\ }{42}$$

one needs to notice that the number to divide by should be increased by 3 for each step (42−39=3); so the preceding number must have been 39−3, or 36. Then,

$$\frac{24}{36} = \frac{2\times12}{3\times12}$$

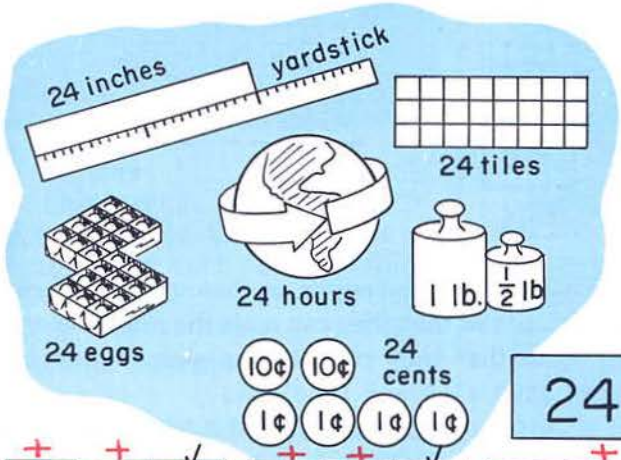
and the missing entries can be found by computing:

$$\frac{2\times12}{3\times12}, \frac{2\times13}{3\times13}, \frac{2\times14}{3\times14}$$



"Have you noticed that the kind of units can change while the number of units remains the same?

"Please complete these records," Mr. Wilson asked.



✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{5}{6}$
6	8	4	24	12	18	3	10	9	20

Find several pairs of machine-settings that produce the same output. Include the pair of machine-settings that uses the smallest possible whole numbers, and record it in the box with the check mark (✓).

24

+	+	✓	+	+	✓	+	✓	+	✓	+	✓	+	✓
$\frac{3}{9}$	$\frac{2}{6}$	$\frac{1}{3}$	$\frac{3}{36}$	$\frac{2}{24}$	$\frac{1}{12}$	$\frac{12}{6}$	$\frac{6}{3}$	$\frac{2}{1}$	$\frac{8}{12}$	$\frac{16}{24}$	$\frac{2}{3}$	$\frac{10}{8}$	$\frac{15}{12}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{3}{2}$
8	8	8	2	2	2	48	48	48	16	16	16	30	30	30	36	36

+	✓	+	✓	+	✓	✓	✓	✓	✓	✓	✓
$\frac{14}{48}$	$\frac{7}{24}$	$\frac{10}{16}$	$\frac{5}{8}$	$\frac{14}{16}$	$\frac{7}{8}$	$\frac{27}{24}$	$\frac{9}{8}$	$\frac{14}{24}$	$\frac{7}{12}$	$\frac{3}{72}$	$\frac{1}{24}$	$\frac{50}{48}$	$\frac{25}{24}$	$\frac{26}{48}$	$\frac{13}{24}$
7	7	15	15	21	21	27	27	14	14	1	1	25	25	13	13

+ MANY POSSIBLE ANSWERS.



5 Dozen Eggs
60 eggs

6 Dimes
60 cents

1 Hour
60 minutes

60

✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\frac{2}{4}$	$\frac{1}{2}$	$\frac{9}{12}$	$\frac{3}{4}$	$\frac{5}{20}$	$\frac{1}{4}$	$\frac{2}{6}$	$\frac{1}{3}$	$\frac{4}{6}$	$\frac{2}{3}$	$\frac{4}{20}$	$\frac{1}{5}$
30	30	45	45	15	15	20	20	40	40	12	12

✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\frac{9}{30}$	$\frac{3}{10}$	$\frac{5}{30}$	$\frac{1}{6}$	$\frac{4}{10}$	$\frac{2}{5}$	$\frac{50}{60}$	$\frac{5}{6}$	$\frac{7}{7}$	$\frac{1}{1}$	$\frac{9}{6}$	$\frac{3}{2}$	$\frac{6}{3}$	$\frac{2}{1}$
18	18	10	10	24	24	50	50	60	60	90	90	120	120



100 cents
100 miles
100 minutes
100 apples
100 pounds
100 feet

100

✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\frac{5}{20}$	$\frac{1}{4}$	$\frac{5}{10}$	$\frac{1}{2}$	$\frac{10}{100}$	$\frac{1}{10}$	$\frac{3}{60}$	$\frac{1}{20}$	$\frac{3}{300}$	$\frac{1}{100}$	$\frac{15}{20}$	$\frac{3}{4}$
25	25	50	50	10	10	5	5	1	1	75	75

✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\frac{42}{60}$	$\frac{7}{10}$	$\frac{45}{100}$	$\frac{9}{20}$	$\frac{6}{4}$	$\frac{3}{2}$	$\frac{21}{300}$	$\frac{7}{100}$	$\frac{10}{5}$	$\frac{2}{1}$	$\frac{115}{500}$	$\frac{23}{100}$	$\frac{8}{10}$	$\frac{4}{5}$
70	70	45	45	150	150	7	7	200	200	23	23	80	80

63 Steps toward generalizing the results of the investigation (Changing the kinds of units used while keeping the numbers of units constant)

Tasks: *Pupils compute equivalent fractions and search for fractions in lowest terms.*

Purpose: To practice computing equivalent fractions and reducing fractions to lowest terms.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: Eventually the pupil must understand that fractions have an existence of their own—as numbers—and are not merely pairs of operations that can be applied to regions or measurements. On pupil page 63 the pupil comes one step closer to this understanding by applying pairs of directions to "24 of anything"—24 inches, 24 tiles, 24 eggs, 24 hours, 24 ounces, 24 cents, all at the same time. We need only a single experiment, perhaps with eggs:

24 eggs $\div 8 = 3$ eggs and 3 eggs $\times 5 = 15$ eggs

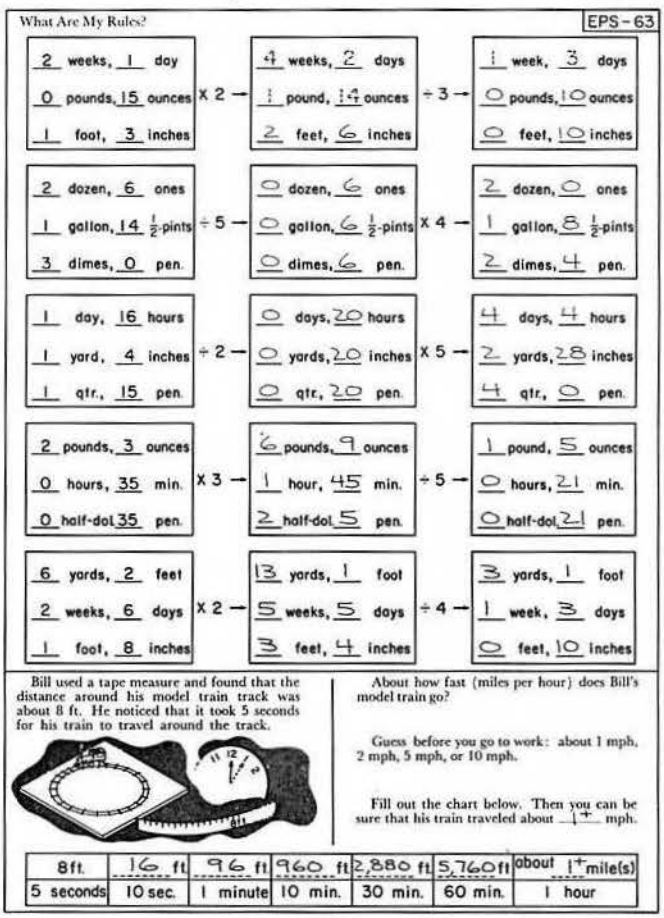
to know that applying ($\times 5$, $\div 8$) to 24 hours will lead us to 15 hours—that applying ($\times 5$, $\div 8$) to 24 of anything will lead us to 15 of whatever we are considering.

Practice sheet 63 introduces another problem in applying pairs of operations. You might ask pupils for various ways of reporting the results of multiplying 2 weeks and 1 day by 2.

2 weeks, 1 day × 2 = 4 weeks, 2 days
= $4\frac{2}{7}$ weeks
= 30 days
= 3 weeks and 9 days

The method of reporting indicated by practice sheet 63 (including the clues given) would call for the first alternative above. It uses the smallest total whole number of the units under consideration. All those given above are correct reports, but only one is appropriate in view of the rules indicated by the format and by the examples given.

The problem at the bottom of practice sheet 63 will probably lead to a healthy discussion. If the model train goes 5,760 feet per hour and there are 5,280 feet in a mile "about 1 mph" or "a little more than 1 mph" are clearly better descriptions than "about 2 mph."



Mr. Wilson describes a more complicated

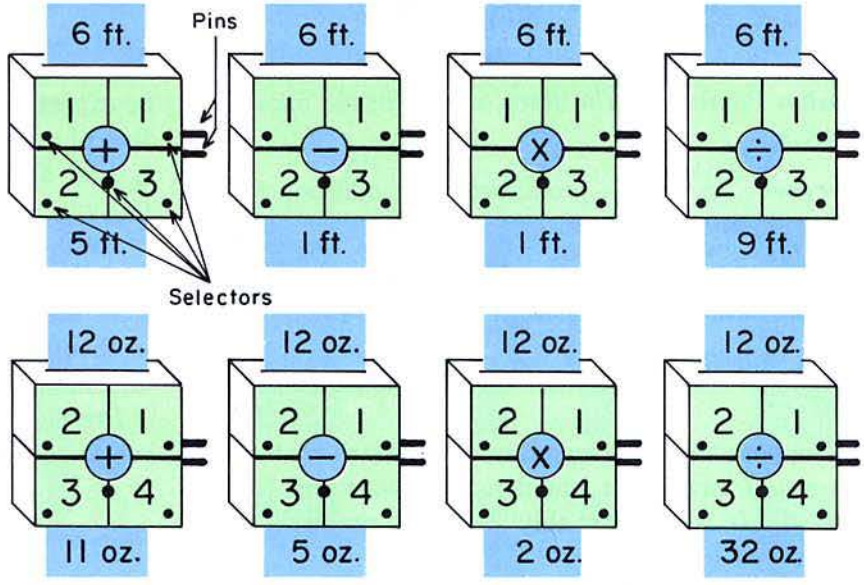
FRACTION MACHINE

designed by Alec Marson.

"This strange machine has 5 selector knobs. One to select a number for each of the four green windows, and another that selects '+,' '−,' '×,' or '÷' for the center indicator.

"The machine has 2 pins on the right."

Samples of Mr. Wilson's records

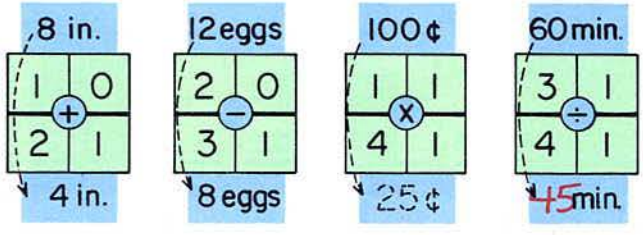


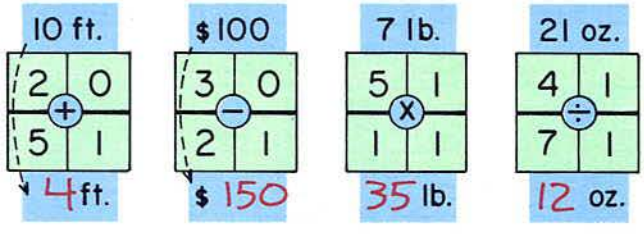
Have you any hunches as to how this new machine yields these outputs?

Mr. Wilson has a few clues.

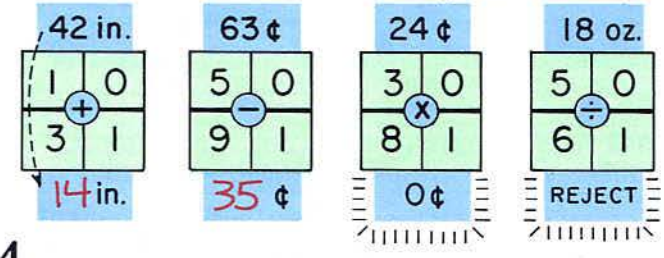
"I found that certain combinations of machine-settings gave surprising results. The outputs are just the same as I would get from the other, simpler machine set according to the left-hand setting on this machine. Here are some results in a kind of shorthand."

(Please fill in the missing entries.)



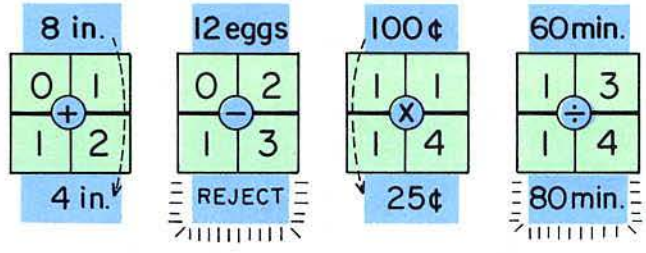


A surprise!



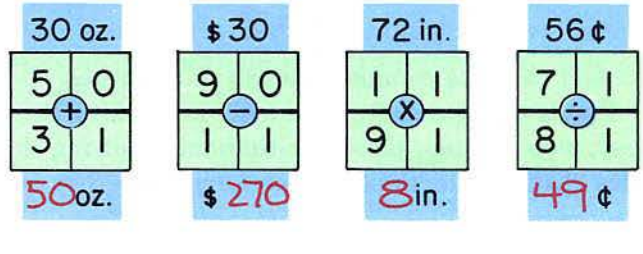
"I certainly did not expect those last two outputs. What happened?

"In my next experiments, I reversed the machine-settings on the right with those on the left."



"Another surprise. In examples set with '+' and '×,' reversing the machine-settings from left to right did not affect the output. So I went back to the first pattern of machine-settings."

(Please fill in the missing entries.)



64 More complicated fraction machines

Tasks: *Pupils read the story and fill in the missing entries.*

Purpose: To lay the groundwork for operations with fractions.

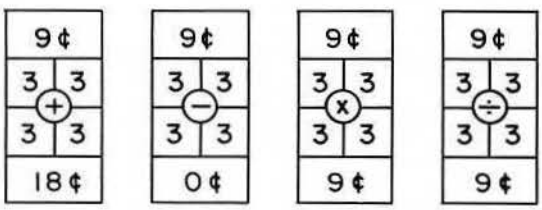
Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Encourage the pupils to speculate about the two sets of four results of Alec's new fraction machine that are given at the top of pupil page 64. How does it work? What rules does it follow?

Some pupils may "know all about it." Some, perhaps, "don't get it." It is probably the case that neither is an accurate report. Those who think they know how to add, subtract, multiply, and divide fractions might have difficulty in explaining the "rejects" that occur; those who don't get it will soon see that this new machine has much in common with the simpler machine.

Hopefully, consideration of pupil page 64 will occur in an atmosphere of exploration and conjecture. The few situations in which pupils are asked to make responses may well be subjects for group discussion. All kinds of explanations for the "surprising" results ought to be considered—none fully rejected, and none accepted "for sure."

What might happen if all the settings were the same—all 3's?

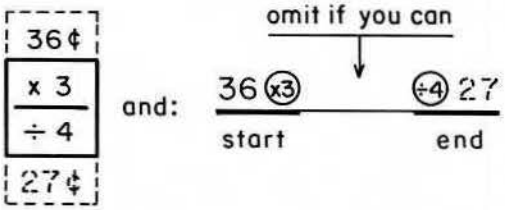


Would the same results hold if all settings were changed to 4? (Yes.)

What if all settings were zero? (All results would be "reject.")

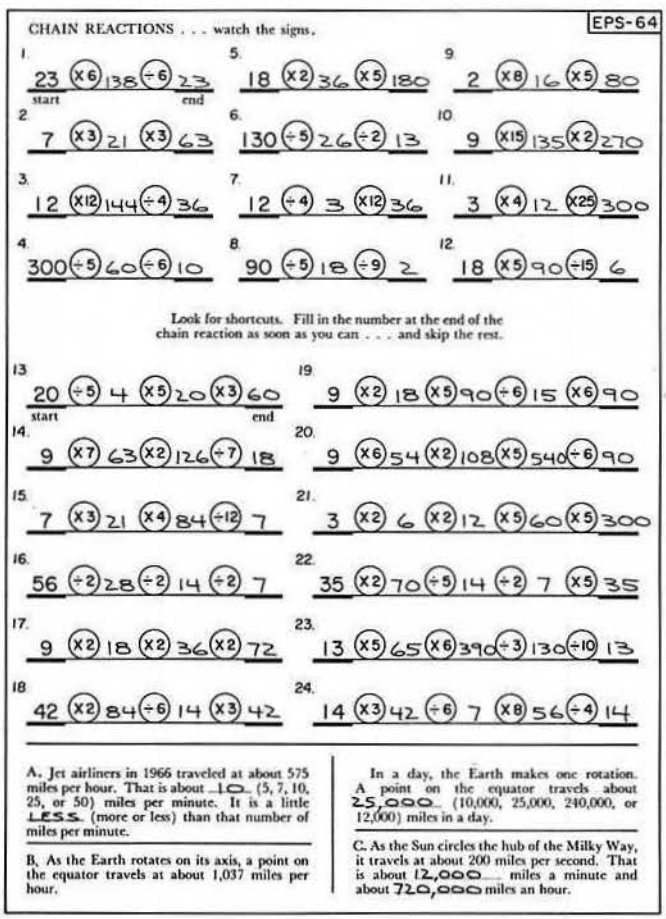
Encourage the pupils to suggest experiments and make guesses about the outcome.

Practice sheet 64 picks up again the idea of a "chain reaction." There is much in common between:



particularly when the pupils are encouraged to look for shortcuts so that they can omit the intermediate step or so that they can perform simpler computations ($36 \div 4 = 9$ and $9 \times 3 = 27$).

The situations discussed at the bottom of the practice sheet again include the common use of the word *about* in referring to speeds and distances. Only common sense can lead the way to interpreting that word—rules will fail. We can say that 575 miles per hour is exactly $9\frac{7}{12}$ miles per minute, but since all we are told is that a jet airliner travels at *about* 575 miles per hour, all conclusions that follow regarding this situation must share the indefiniteness implied by *about*.


CHAIN REACTIONS . . . watch the signs. EPS-64

1. 23 ×6 138 ÷6 23 (start, end)
2. 7 ×3 21 ×3 63
3. 12 ×12 144 ÷4 36
4. 300 ÷5 60 ÷6 10
5. 18 ×2 36 ×5 180
6. 130 ÷5 26 ÷2 13
7. 12 ÷4 3 ×12 36
8. 90 ÷5 18 ÷9 2
9. 2 ×8 16 ×5 80
10. 9 ×15 135 ×2 270
11. 3 ×4 12 ×25 300
12. 18 ×5 90 ÷15 6

Look for shortcuts. Fill in the number at the end of the chain reaction as soon as you can . . . and skip the rest.

13. 20 ÷5 4 ×5 20 ×3 60 (start, end)
14. 9 ×7 63 ×2 126 ÷7 18
15. 7 ×3 21 ×4 84 ÷12 7
16. 56 ÷2 28 ÷2 14 ÷2 7
17. 9 ×2 18 ×2 36 ×2 72
18. 42 ×2 84 ÷6 14 ×3 42
19. 9 ×2 18 ×5 90 ÷6 15 ×6 90
20. 9 ×6 54 ×2 108 ×5 540 ÷6 90
21. 3 ×2 6 ×2 12 ×5 60 ×5 300
22. 35 ×2 70 ÷5 14 ÷2 7 ×5 35
23. 13 ×5 65 ×6 390 ÷3 130 ÷10 13
24. 14 ×3 42 ÷6 7 ×8 56 ÷4 14

A. Jet airliners in 1966 traveled at about 575 miles per hour. That is about 10 (5, 7, 10, 25, or 50) miles per minute. It is a little LESS (more or less) than that number of miles per minute.

B. As the Earth rotates on its axis, a point on the equator travels at about 1,037 miles per hour.

In a day, the Earth makes one rotation. A point on the equator travels about 25,000 (10,000, 25,000, 240,000, or 12,000) miles in a day.

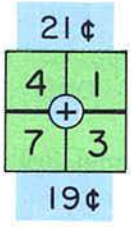
C. As the Sun circles the hub of the Milky Way, it travels at about 200 miles per second. That is about 12,000 miles a minute and about 720,000 miles an hour.

Mr. Wilson described his next series of experiments.

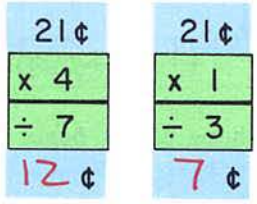
"I compared the settings on the new machine and on pairs of the simple machines . . . using the ' + ' and ' − ' center settings."

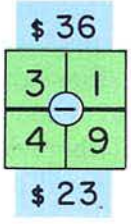
(Please fill in the missing entries.)

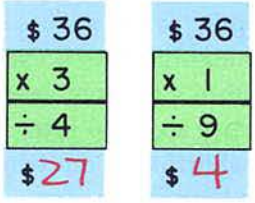
New machine



A pair of hookups of old machines



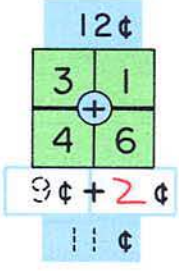


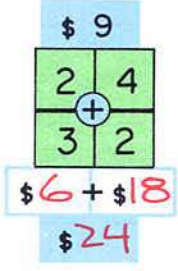


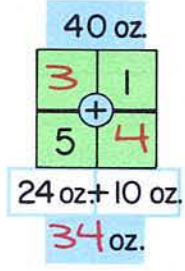
"Did you notice anything about those results?

"I did, and I kept records of the next experiments in this way."

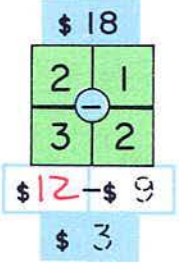
Addition

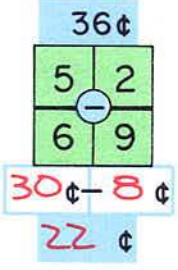


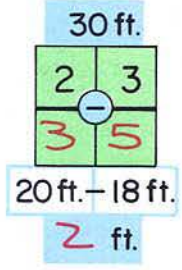




Subtraction





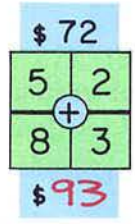


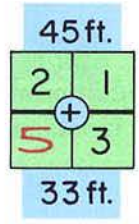
"I also tried the same approach with the ' × ' and ' ÷ ' settings. I'll discuss my results later."

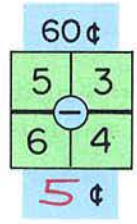
"Now I could know ahead of time the results of Alec's machines when they were set for addition or subtraction.

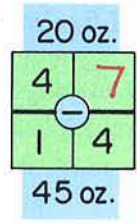
"Here are some of my notes."

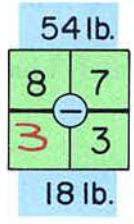
(Please fill in the missing entries.)

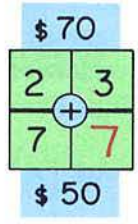


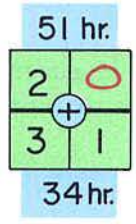


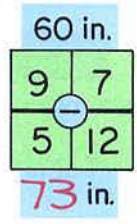






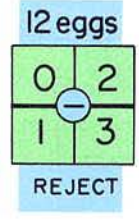




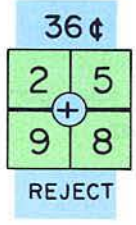


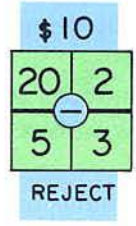
"I still needed to find why the machine rejected one of my earlier experiments.

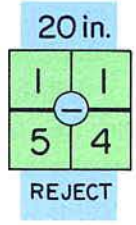
"Why?" THE MACHINE CAN'T HANDLE 0 EGGS − 8 EGGS.

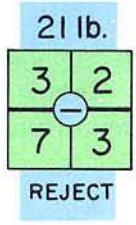


"Here are other examples in which the output was 'REJECT.'"









"Some were rejected for one reason and some for another.

"It seems that the dividing mechanism in Alec's machines can handle only counting numbers — it is not equipped to respond to '36¢ ÷ 8' nor to '4 − 5.'

"Now I'm ready to tackle the problem of those ' × ' and ' ÷ ' machine-settings."

65 Exploring the addition and subtraction settings on Alec's new machines

Tasks: *Pupils read the story and fill in the missing entries.*

Purpose: To lay the groundwork for addition and subtraction of fractions.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Experiments with Alec's machine progress.

Suppose that one wondered how much material he would have if he added $\frac{5}{6}$ of a yard and $\frac{3}{4}$ of a yard. He would probably go at the problem this way:

a. there are 36 inches in a yard

b. $\frac{1}{6}$ of a yard is 6 inches ($\div 6$)

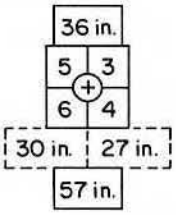
c. $\frac{5}{6}$ of a yard must be 6 in. $\times 5$, or 30 inches ($\times 5$)

d. $\frac{1}{4}$ of a yard is 9 inches ($\div 4$)

e. $\frac{3}{4}$ of a yard must be 9 in. $\times 3$, or 27 inches ($\times 3$)

f. 30 inches + 27 inches = 57 inches

A record of Alec's machine, set to handle that problem (and with Mr. Wilson's record of intermediate results given), would look like this:



Thus Alec's machine approaches the problem step by step in exactly the common-sense way some people think about such questions in everyday life.

Very few would take the following sophisticated course of action:

$$\frac{5}{6} + \frac{3}{4} = \frac{10}{12} + \frac{9}{12} = \frac{19}{12}$$

$$\frac{19}{\cancel{12}} \times \frac{3}{\cancel{36}} = 57$$

to find that the combined length is 57 inches.

Pupils need extensive experience in thinking their way, step by step, through situations involving the idea of adding fractions *of something* before they can be expected to attach much meaning to the abstraction of adding the abstraction $\frac{3}{4}$ to the abstraction $\frac{5}{6}$ and asserting that the sum is the abstraction $\frac{19}{12}$.

Keep in mind as we proceed that we are trying first to help pupils understand fully the way fractions are used in everyday, common-sense patterns of thought—reserving, for a while, any consideration of fractions as abstractions.

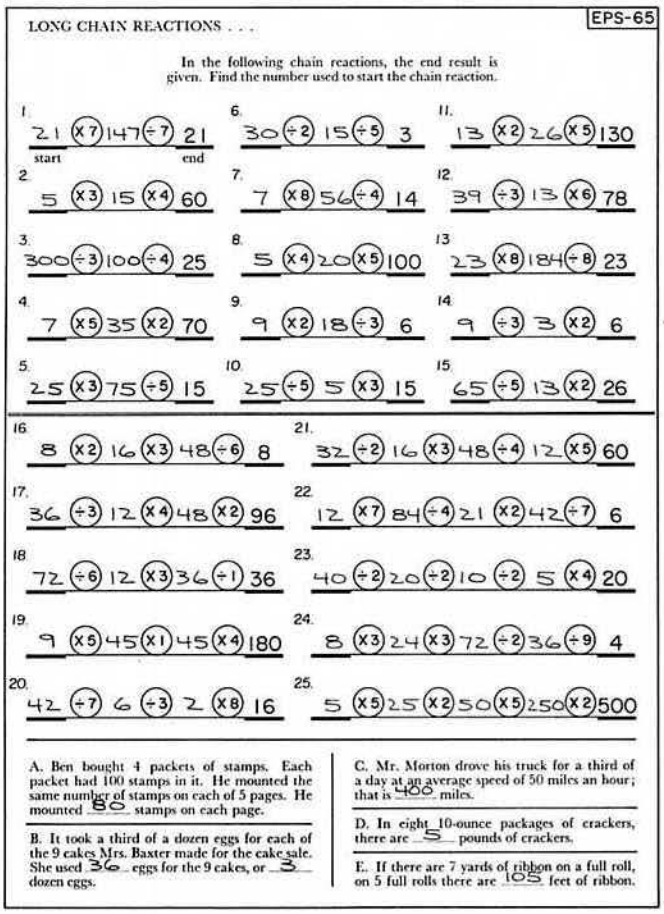
Practice sheet 65 asks pupils to reconstruct chain reactions. In effect, the exercise:

7 (×8) 56 (÷4) 14

start end

poses this situation: 14 is the result of dividing by 4, and how can you undo such an act? $\times 4$ undoes $\div 4$. So, 14×4, or 56, was divided by 4 to produce 14. Next, 56 is the result of multiplying by 8, and you can undo this by dividing by 8.

This activity helps develop increased familiarity with the basic idea of pairs of operations and plants seeds we hope to nurture later when we consider fractions in the role of divisors.



LONG CHAIN REACTIONS . . . EPS-65

In the following chain reactions, the end result is given. Find the number used to start the chain reaction.

1. 21 (×7) 147 (÷7) 21 (start, end)
2. 5 (×3) 15 (×4) 60
3. 300 (÷3) 100 (÷4) 25
4. 7 (×5) 35 (×2) 70
5. 25 (×3) 75 (÷5) 15
6. 30 (÷2) 15 (÷5) 3
7. 7 (×8) 56 (÷4) 14
8. 5 (×4) 20 (×5) 100
9. 9 (×2) 18 (÷3) 6
10. 25 (÷5) 5 (×3) 15
11. 13 (×2) 26 (×5) 130
12. 39 (÷3) 13 (×6) 78
13. 23 (×8) 184 (÷8) 23
14. 9 (÷3) 3 (×2) 6
15. 65 (÷5) 13 (×2) 26
16. 8 (×2) 16 (×3) 48 (÷6) 8
17. 36 (÷3) 12 (×4) 48 (×2) 96
18. 72 (÷6) 12 (×3) 36 (÷1) 36
19. 9 (×5) 45 (×1) 45 (×4) 180
20. 42 (÷7) 6 (÷3) 2 (×8) 16
21. 32 (÷2) 16 (×3) 48 (÷4) 12 (×5) 60
22. 12 (×7) 84 (÷4) 21 (×2) 42 (÷7) 6
23. 40 (÷2) 20 (÷2) 10 (÷2) 5 (×4) 20
24. 8 (×3) 24 (×3) 72 (÷2) 36 (÷9) 4
25. 5 (×5) 25 (×2) 50 (×5) 250 (×2) 500

A. Ben bought 4 packets of stamps. Each packet had 100 stamps in it. He mounted the same number of stamps on each of 5 pages. He mounted 80 stamps on each page.

B. It took a third of a dozen eggs for each of the 9 cakes Mrs. Baxter made for the cake sale. She used 36 eggs for the 9 cakes, or 3 dozen eggs.

C. Mr. Morton drove his truck for a third of a day at an average speed of 50 miles an hour; that is 400 miles.

D. In eight 10-ounce packages of crackers, there are 5 pounds of crackers.

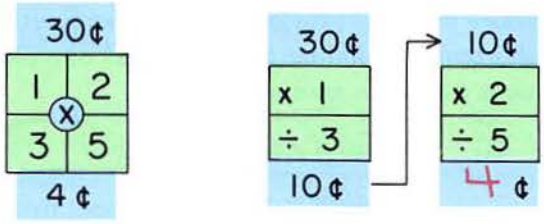
E. If there are 7 yards of ribbon on a full roll, on 5 full rolls there are 105 feet of ribbon.

"×" "÷"

More Experiments with Alec's Fraction Machine

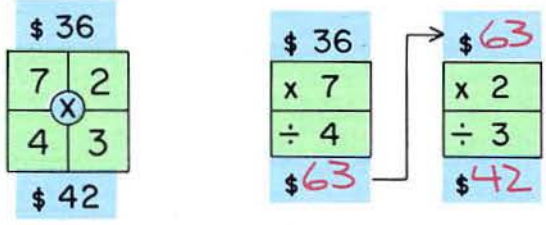
"I tried comparing the output of Alec's new machine set at '×' with a pair of Alec's simpler machine hookups . . . just as I had with the new machine set at '+' and '−.' For a long time, I found no clues.

"Then, almost by accident, I found a most surprising result. I kept a record of it.

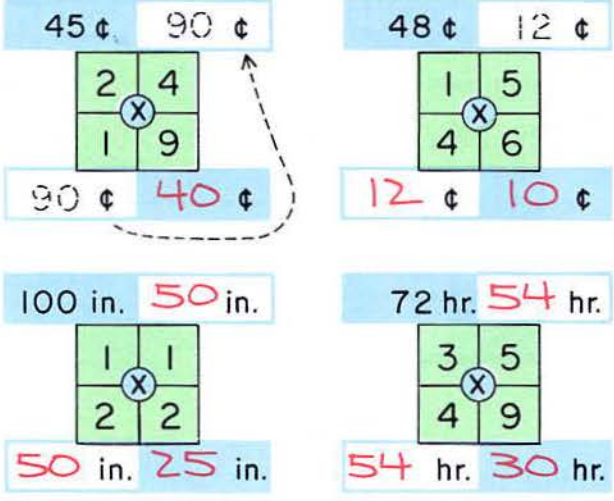


"I took the output from the first simple fraction machine and used it as an input for a second one.

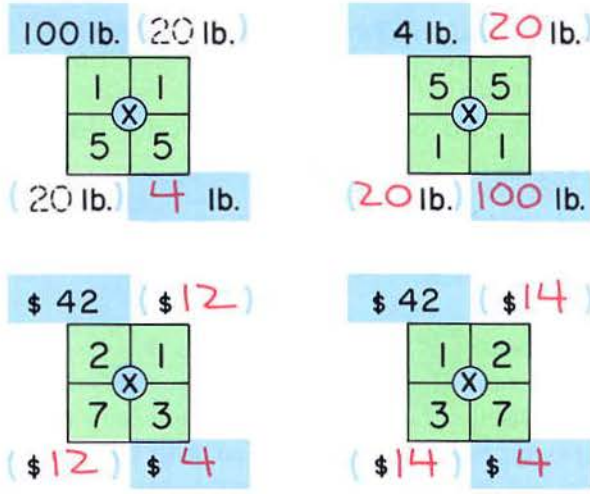
"So I tried again." *(Please fill the blanks.)*



"As I made more experiments, I kept records in this way:

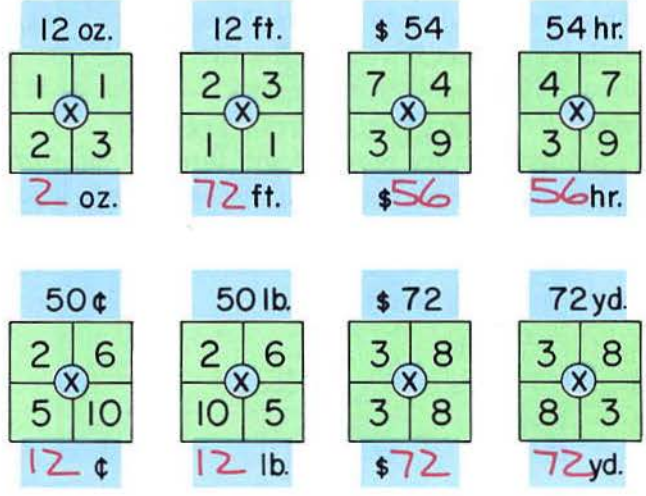


"Then, for a few experiments, I kept my records this way:

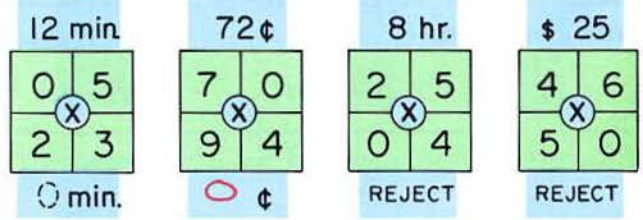


"Finally, I left out the record of the first output that was also the input for the second step. This enabled me to keep more records in less space."

(Please fill the blanks.)



"I wondered what would happen if I used a zero in each group of settings.



"I should have expected those rejects — it makes no sense to divide by 0."

66 Exploring the multiplication setting on Alec's machines

Tasks: *Pupils read the story and fill in the missing entries.*

Purpose: To lay the groundwork for multiplication of fractions.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: With pupil page 66, the investigation temporarily shifts from + and — settings to × and ÷ settings.

In everyday language, we understand clearly what we mean when we write things like:

$\frac{1}{2}$ of 60 minutes is 30 minutes (×1, ÷2)

$\frac{4}{5}$ of 60 minutes is 48 minutes (×4, ÷5)

30 min. + 48 min. = 78 min.

Pupils might expect Alec's machine, with a × setting in the center, to produce a result for the following question:

30 min. × 48 min. = ?

But, of course, this question has no useful answer. Alec certainly knew this; so he constructed a machine that follows a very useful notion of multiplication.

Alec's machine finds the following situation most reasonable:

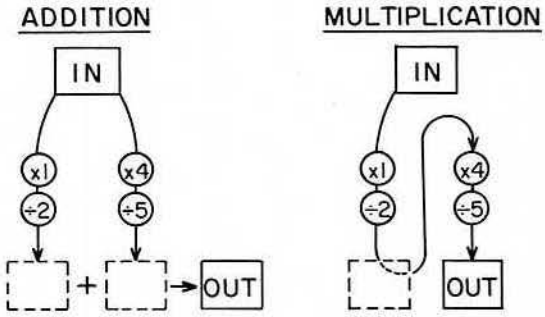
"Mary had a half-hour of free time; she spent four-fifths of her free time writing a letter.

$\frac{1}{2}$ of 60 minutes is 30 minutes (×1, ÷2)

$\frac{4}{5}$ of 30 minutes is 24 minutes (÷5, ×4)

So, Mary spent 24 minutes writing a letter."

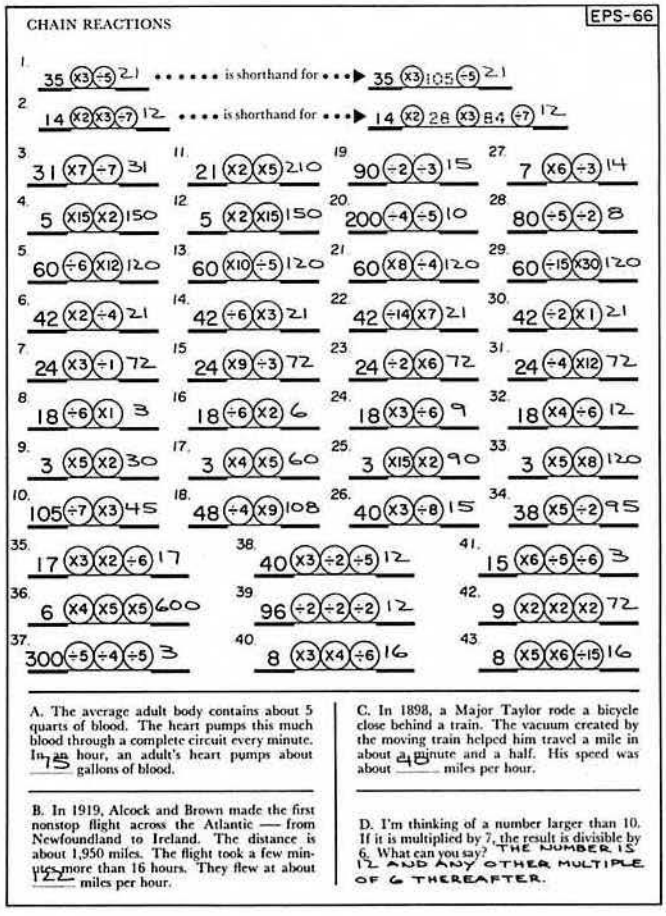
It's as if Alec imagined two basic routes in his machines between input and output—one capturing the idea of addition and the other capturing the idea of multiplication.



We hope pupils *begin* to have a solid feeling of the very basic difference between the ideas of addition and multiplication.

Notice that the examples at the top of the right-hand column show that Alec's machine has a means for undoing what it does—an input of 100 lbs. may lead to an output of 4 lbs.; but, with other settings, an input of 4 lbs. may lead to an output of 100 lbs.

Practice sheet 66 introduces a slight abbreviation in the shorthand, omitting from the record one or more intermediate steps.


CHAIN REACTIONS EPS-66

1. 35 (×3)(÷5) 21 ····· is shorthand for ···▶ 35 (×3) 105 (÷5) 21
2. 14 (×2)(×3)(÷7) 12 ···· is shorthand for ···▶ 14 (×2) 28 (×3) 84 (÷7) 12

3. 31 (×7)(÷7) 31
4. 5 (×15)(×2) 150
5. 60 (÷6)(×12) 120
6. 42 (×2)(÷4) 21
7. 24 (×3)(÷1) 72
8. 18 (÷6)(×1) 3
9. 3 (×5)(×2) 30
10. 105 (÷7)(×3) 45
11. 21 (×2)(×5) 210
12. 5 (×2)(×15) 150
13. 60 (×10)(÷5) 120
14. 42 (÷6)(×3) 21
15. 24 (×9)(÷3) 72
16. 18 (÷6)(×2) 6
17. 3 (×4)(×5) 60
18. 48 (÷4)(×9) 108
19. 90 (÷2)(÷3) 15
20. 200 (÷4)(÷5) 10
21. 60 (×8)(÷4) 120
22. 42 (÷14)(×7) 21
23. 24 (÷2)(×6) 72
24. 18 (×3)(÷6) 9
25. 3 (×15)(×2) 90
26. 40 (×3)(÷8) 15
27. 7 (×6)(÷3) 14
28. 80 (÷5)(÷2) 8
29. 60 (÷15)(×30) 120
30. 42 (÷2)(×1) 21
31. 24 (÷4)(×12) 72
32. 18 (×4)(÷6) 12
33. 3 (×5)(×8) 120
34. 38 (×5)(÷2) 95
35. 17 (×3)(×2)(÷6) 17
36. 6 (×4)(×5)(×5) 600
37. 300 (÷5)(÷4)(÷5) 3
38. 40 (×3)(÷2)(÷5) 12
39. 96 (÷2)(÷2)(÷2) 12
40. 8 (×3)(×4)(÷6) 16
41. 15 (×6)(÷5)(÷6) 3
42. 9 (×2)(×2)(×2) 72
43. 8 (×5)(×6)(÷15) 16

A. The average adult body contains about 5 quarts of blood. The heart pumps this much blood through a complete circuit every minute. In $\frac{1}{2}$ hour, an adult's heart pumps about 15 gallons of blood.

B. In 1919, Alcock and Brown made the first nonstop flight across the Atlantic — from Newfoundland to Ireland. The distance is about 1,950 miles. The flight took a few minutes more than 16 hours. They flew at about 122 miles per hour.

C. In 1898, a Major Taylor rode a bicycle close behind a train. The vacuum created by the moving train helped him travel a mile in about $\frac{1}{2}$ minute and a half. His speed was about ____ miles per hour.

D. I'm thinking of a number larger than 10. If it is multiplied by 7, the result is divisible by 6. What can you say? THE NUMBER IS 12 AND ANY OTHER MULTIPLE OF 6 THEREAFTER.

"The key clue which led to my discovering the secret of Alec's fraction machine with a '÷' setting was the fact that we cannot find any reasonable meaning for division by zero.

"Sometimes, people want to say that dividing a measurement by zero does not change the measurement. That leads to a trap:

If 21¢ ÷ 3 = 7¢ then 7¢ x 3 = 21¢
If 21¢ ÷ 0 = 21¢ then 21¢ x 0 = 21¢

and that is not true!

"Sometimes, people want to say that any amount divided by zero is 'infinity' — ∞ in shorthand. Whatever they mean by 'infinity,' they fall into this trap:

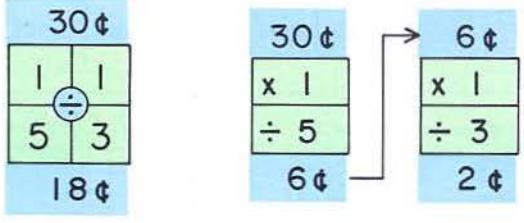
25 miles ÷ 0 = ∞ and 0 x ∞ = 25 miles
9 miles ÷ 0 = ∞ and 0 x ∞ = 9 miles

so 25 miles = 9 miles. (False!)

"Mathematicians have never discovered a reasonable meaning for division by zero. So they reject the idea . . . as Alec's machine does."

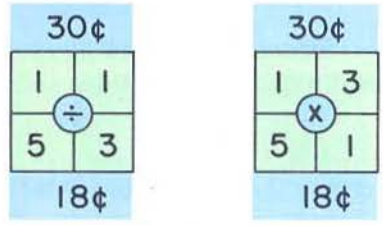
Reversing the instructions!

"I tried the experiment of using the output of one of Alec's machines as input for another — as I had with the '×' settings:

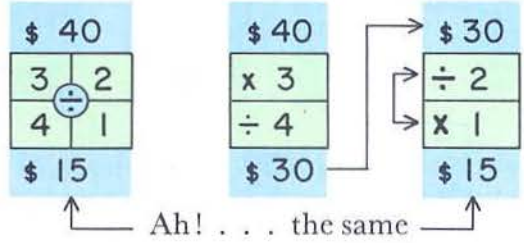


"If Alec's second machine had only multiplied by 3 instead of divided by 3 and divided by 1 instead of multiplied by 1!

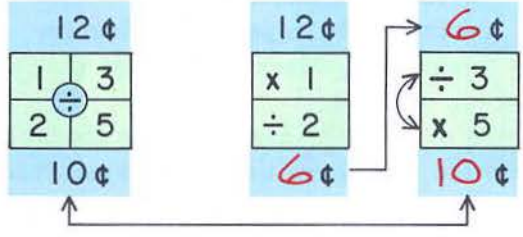
"The fraction machine with a '÷' setting acts just like a fraction machine with a '×' setting but with the numbers on the right interchanged.



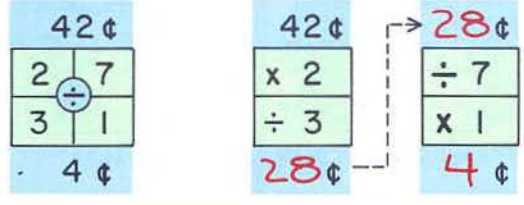
"Maybe I should try switching the '×' and '÷' on Alec's second machine hookup. I tried a new example involving the fraction machine with a '÷' setting:

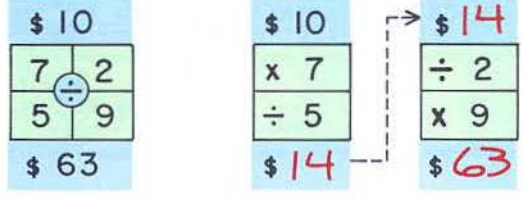


"I tried another example:



"Try again!"





67 Exploring the division setting on Alec's machines

Tasks: *Pupils read the story and fill in the missing entries.*

Purpose: To lay the groundwork for division of fractions.

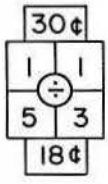
Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: What could the direction "$\div 0$" possibly mean? Consider every idea suggested by the class. Discuss the implications of each proposal. Perhaps you may take the position with the class that it would be particularly unfortunate if no reasonable interpretation could be found. . . . Then reluctantly conclude at the end of the discussion that your worst fears have been realized—because no one yet has been able to find any meaning that wouldn't upset all the rest of arithmetic.

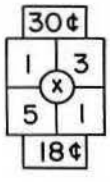
For example, what could "a dozen eggs $\div$ 0" mean? Could it mean simply leaving the dozen eggs alone? This might lead us to say that 12 eggs $\div 0 = 12$ eggs. If that were true, it would follow that 12 eggs $\times 0 = 12$ eggs. But we know that 12 eggs $\times 0 = 0$ eggs. So if 12 eggs $\div 0 = 12$ eggs then 12 eggs $= 0$ eggs!

Similarly, every proposal will threaten disaster—so, we *must simply agree* that we shall have to ignore the instruction "$\div 0$."

What problem is Alec's fraction machine solving in the following case?



Here is one possible problem. "Mary saved $\frac{1}{5}$ of her 30¢ allowance. That amounted to $\frac{1}{3}$ of Bill's allowance. How much was Bill's allowance?" But this problem can be worked by Alec's fraction machine with a $\times$ setting:



It seems that a $\div$ setting has the effect on the settings that follow it of reversing the agreement about the operations—that the number above the cross line is a number to multiply by and the number below is a number to divide by. One way to undo the pair of operations ($\times 3$, $\div 5$) is to apply the pair ($\div 3$, $\times 5$).

30 (×3)(÷5) 18 and 18 (×3)(÷5) 30

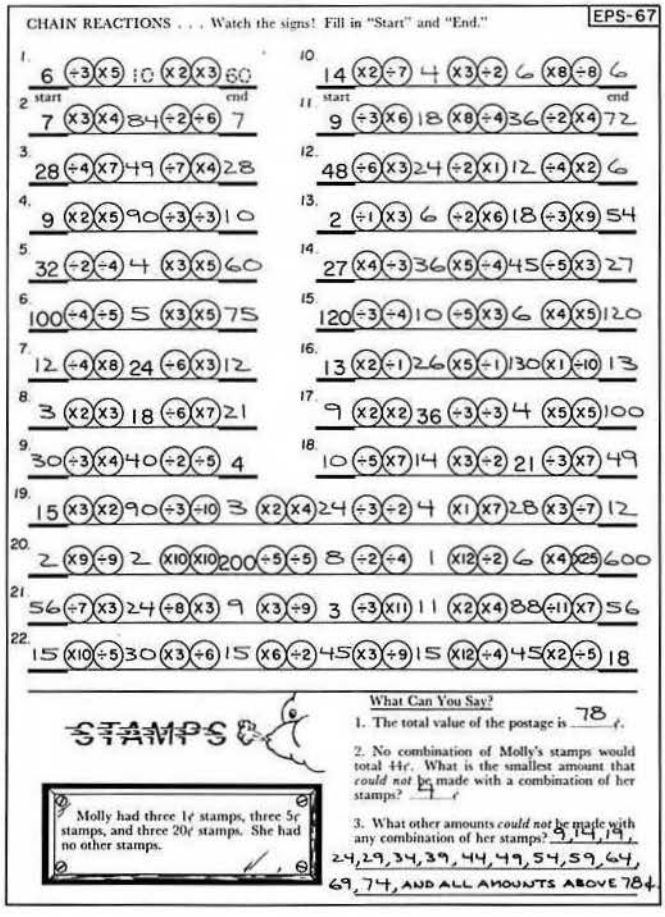
It is on this notion that the meaning of dividing by a fraction will be built.

Practice sheet 67 provides many opportunities to find shortcuts, including this idea that one pair of directions can undo another.

28 (÷4)(×7) 49 (×4)(÷7) 28

12 (÷4)(×8) 24 (÷6)(×3) 12

Every shortcut the pupils notice is a discovery that deepens their understanding of fractions.


CHAIN REACTIONS . . . Watch the signs! Fill in "Start" and "End." EPS-67

1. 6 (÷3)(×5) 10 (×2)(×3) 60
2. start 7 (×3)(×4) 84 (÷2)(÷6) 7 end
3. 28 (÷4)(×7) 49 (÷7)(×4) 28
4. 9 (×2)(×5) 90 (÷3)(÷3) 10
5. 32 (÷2)(÷4) 4 (×3)(×5) 60
6. 100 (÷4)(÷5) 5 (×3)(×5) 75
7. 12 (÷4)(×8) 24 (÷6)(×3) 12
8. 3 (×2)(×3) 18 (÷6)(×7) 21
9. 30 (÷3)(×4) 40 (÷2)(÷5) 4
10. 14 (×2)(÷7) 4 (×3)(÷2) 6 (×8)(÷8) 6
11. start 9 (÷3)(×6) 18 (×8)(÷4) 36 (÷2)(×4) 72 end
12. 48 (÷6)(×3) 24 (÷2)(×1) 12 (÷4)(×2) 6
13. 2 (÷1)(×3) 6 (÷2)(×6) 18 (÷3)(×9) 54
14. 27 (×4)(÷3) 36 (×5)(÷4) 45 (÷5)(×3) 27
15. 120 (÷3)(÷4) 10 (÷5)(×3) 6 (×4)(×5) 120
16. 13 (×2)(÷1) 26 (×5)(÷1) 130 (×1)(÷10) 13
17. 9 (×2)(×2) 36 (÷3)(÷3) 4 (×5)(×5) 100
18. 10 (÷5)(×7) 14 (×3)(÷2) 21 (÷3)(×7) 49
19. 15 (×3)(×2) 90 (÷3)(÷10) 3 (×2)(×4) 24 (÷3)(÷2) 4 (×1)(×7) 28 (×3)(÷7) 12
20. 2 (×9)(÷9) 2 (×10)(×10) 200 (÷5)(÷5) 8 (÷2)(÷4) 1 (×12)(÷2) 6 (×4)(×25) 600
21. 56 (÷7)(×3) 24 (÷8)(×3) 9 (×3)(÷9) 3 (÷3)(×11) 11 (×2)(×4) 88 (÷11)(×7) 56
22. 15 (×10)(÷5) 30 (×3)(÷6) 15 (×6)(÷2) 45 (×3)(÷9) 15 (×12)(÷4) 45 (×2)(÷5) 18

STAMPS

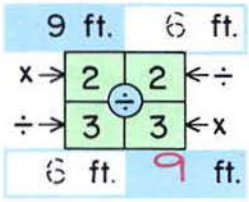
Molly had three 1¢ stamps, three 5¢ stamps, and three 20¢ stamps. She had no other stamps.

What Can You Say?

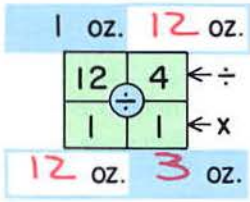
1. The total value of the postage is 78 ¢.
2. No combination of Molly's stamps would total 44¢. What is the smallest amount that *could not* be made with a combination of her stamps? 1 ¢
3. What other amounts *could not* be made with any combination of her stamps? 3, 14, 19, 24, 29, 34, 39, 44, 49, 54, 59, 64, 69, 74, AND ALL AMOUNTS ABOVE 78¢

"Here are records of more experiments with the '÷' setting on Alec's fraction machine." *(Please fill in the missing entries.)*

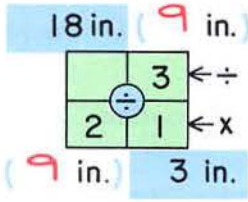
1.



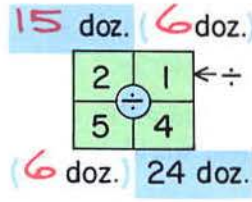
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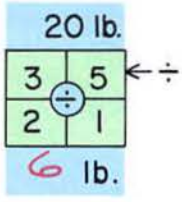
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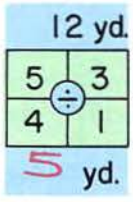
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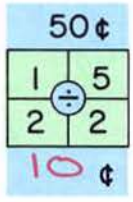
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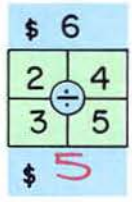
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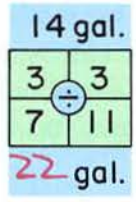
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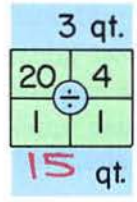
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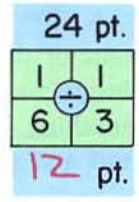
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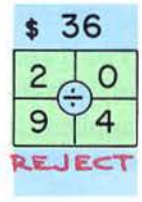
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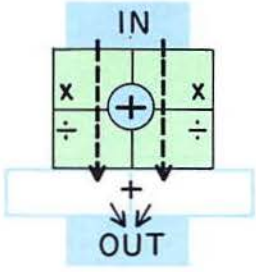
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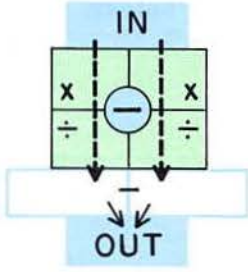


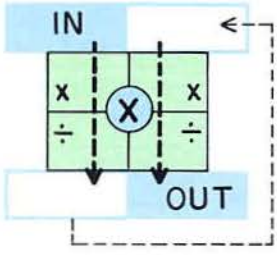
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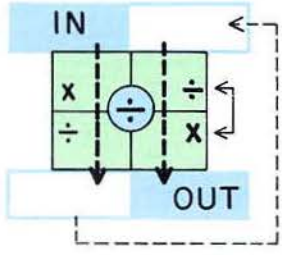


"I made some sketches to help me remember the way Alec's machines work.





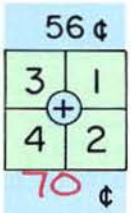




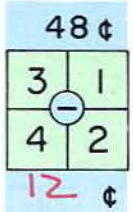
"Here are samples from the records of my experiments. WATCH THE SIGNS (the center settings)."

(Please fill in the missing entries.)

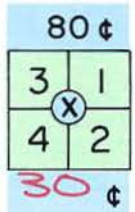
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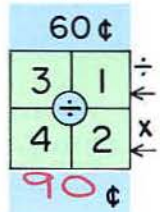
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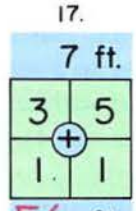
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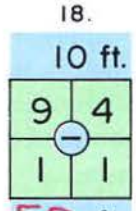
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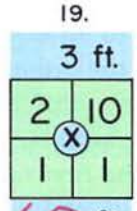
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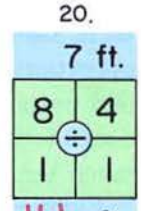
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19.

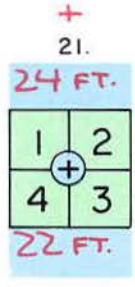


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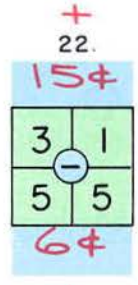


"In the following examples, use any inputs that Alec's fraction machine will not REJECT."

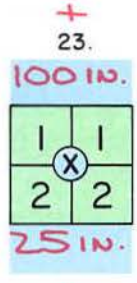
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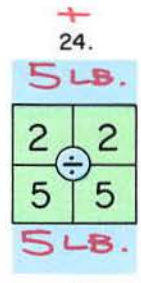
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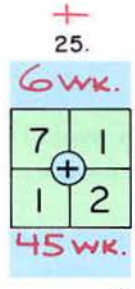
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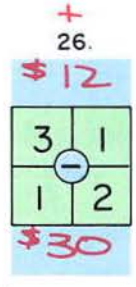
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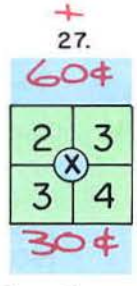
25.



26.



27.

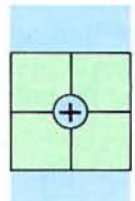


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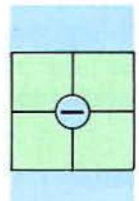


(Make up examples of your own.) +

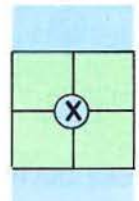
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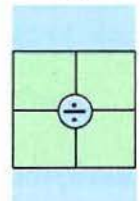
30.



31.



32.



+ MANY POSSIBLE ANSWERS.

68 Exploring all four settings (+, -, x, /) on Alec's machines

Tasks: *Pupils read the story and fill in the missing entries.*

Purpose: To lay the groundwork for division of fractions, and to compare all four operations with fractions.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: The first set of exercises (1-12) is provided to help clarify the way in which Alec's machine seems to proceed when the central setting is ÷, calling for division by a pair of operations; the ÷ setting simply reverses the order of operation signs in the agreement that

$$\boxed{\frac{2}{5}} \text{ means } \frac{\times 2}{\div 5} \text{ or } \times 2, \div 5.$$

Thus,

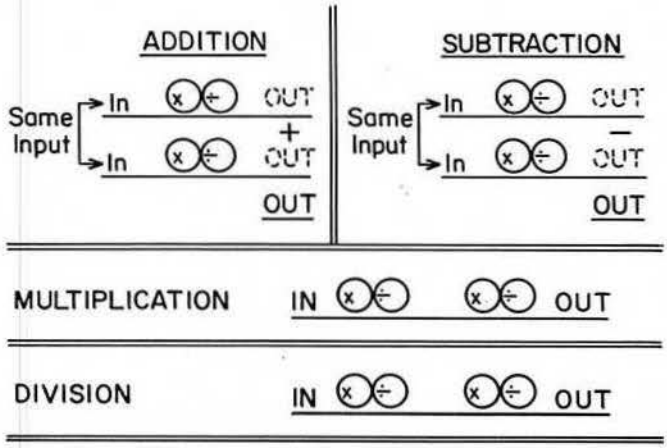
$$\boxed{\begin{matrix}\div 2\\ \times 5\end{matrix}} \text{ means } \frac{\div 2}{\times 5} \text{ or } \div 2, \times 5$$

In this way, ÷ is the signal to undo what would have been done if × had preceded it instead.

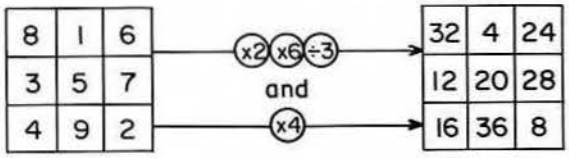
We shall return to this idea later.

The remainder of the page is a summary of the results produced by Alec's machine as the various settings are changed.

A series of sketches—flow charts—is given to help visualize the procedures the machine seems to follow. In the shorthand of the practice sheets, the general plans might look like this:



Practice sheet 68 considers directions such as ×2 and ÷6 as they might be applied to Magic Squares. Pupils are encouraged to look for shortcuts. Some may notice that the first chain reaction could avoid the intermediate steps:



Aren't you tempted to say that

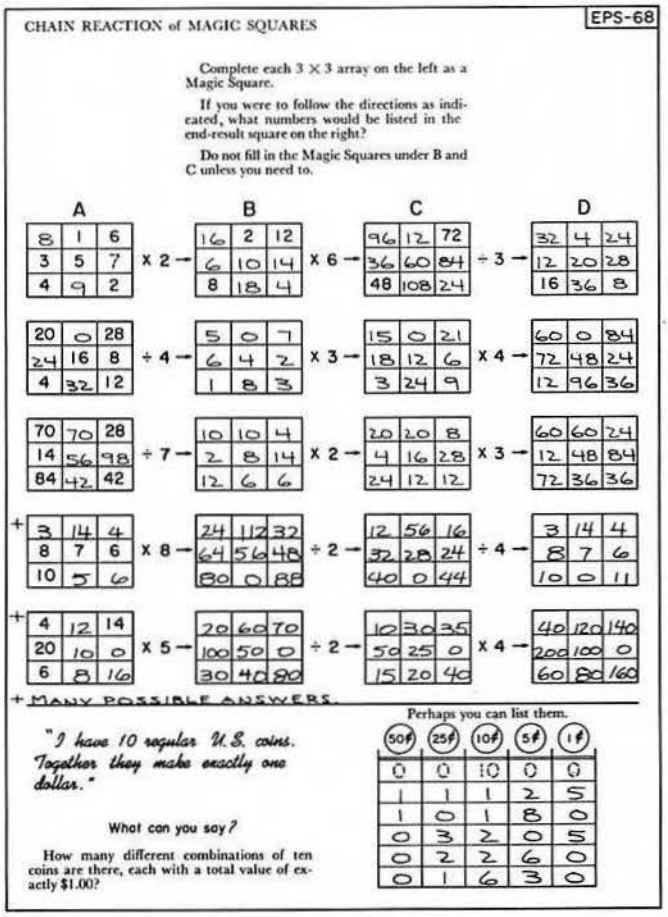
$$\text{x2 x6 ÷3} = \text{x4 ÷1}\ ?$$

And this is the notion we shall develop with investigation of Alec's Simplifier, introduced on pupil page 71.

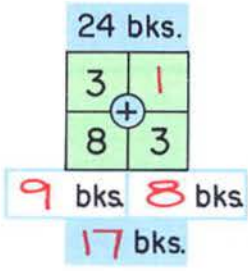
Other chain reactions on practice sheet 68 may suggest the idea that

$$\text{÷4 x3 x4} = \text{x3 ÷1} \qquad \text{÷7 x2 x3} = \text{÷7 x6} = \text{x6 ÷7}$$

$$\text{x8 ÷2 ÷4} = \text{x1 ÷1} \qquad \text{x5 ÷2 x4} = \text{x10 ÷1}$$



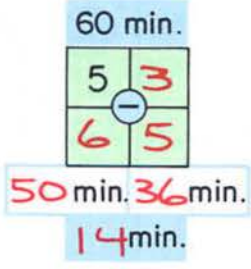
A.



Alice and Mary had agreed to collect twenty-four used books for the school book sale. Alice had collected three-eighths of that number of books. Mary had collected one-third of that number of books.

Alice had collected 9 books. Mary had collected 8 books. Together, they had collected 17 books. The number of books they both collected is 7 less than their goal.

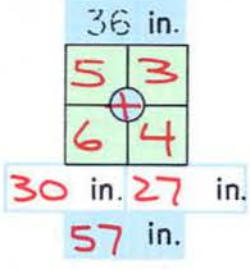
B.



One day Terry spent five-sixths of an hour on his homework and practiced playing his violin for three-fifths of an hour.

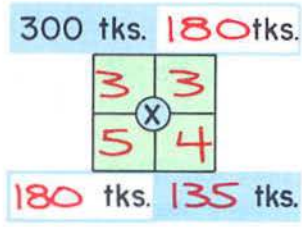
He spent 14 minutes more on his homework than on his violin practice.

C.



Five-sixths of a yard added to three-fourths of a yard is 57 inches; that is LESS (more or less) than five feet.

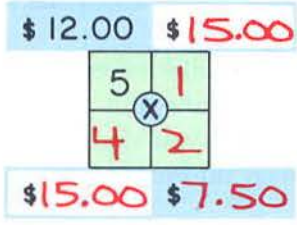
D.



The Drama Club had 300 tickets printed. Three-fifths of those printed were sold Of those sold, three-fourths were adult tickets.

There were 135 adult tickets sold.

E.

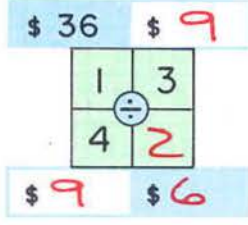


A group of four boys worked for five days. As a group, they received $12.00 a day. The boys divided their money evenly.

Jack spent half of his earnings. He must have spent $ 7.50.

If Jack had spent a third of his earnings instead of a half, he would have spent LESS (more or less) than he actually spent.

F.



Bill earned thirty-six dollars. He put all but one-fourth of it in the bank. With the amount he had left, he could buy exactly three of the equally priced model rockets he wanted. He bought two models.

Bill spent a total of $ 6 for the two rockets.

69-70 Using Alec's machines to help solve word problems

Tasks: *Pupils read problems A-F and fill in the missing entries.*

Purpose: To use the "machine procedures" to solve real world problems.

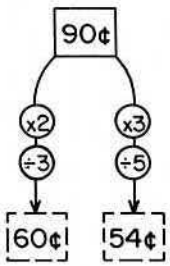
Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: One of the most important outcomes of thinking about fractions as pairs of operations applied to something is that applications are so easy to work out and to understand.

Addition and Subtraction

Consider any amount. Apply a pair of operations to it, leading to an output. Apply a second pair of operations to the original amount, leading to a second output. In addition, we are concerned with the sum of these outputs. In subtraction, we ask how much larger the first output is.

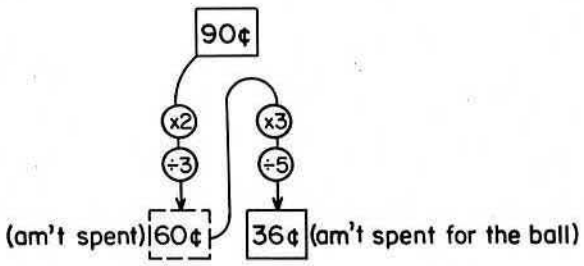
Example: Bill and Jane each get an allowance of 90¢ a week. Bill spent $\frac{2}{3}$ of his weekly allowance and Jane spent $\frac{3}{5}$ of hers. Together, how much did they spend? (Addition) How much more did Bill spend? (Subtraction)



Together, they spent 60¢ + 54¢, or $1.14. Bill spent 60¢ − 54¢, or 6¢ more.

Multiplication

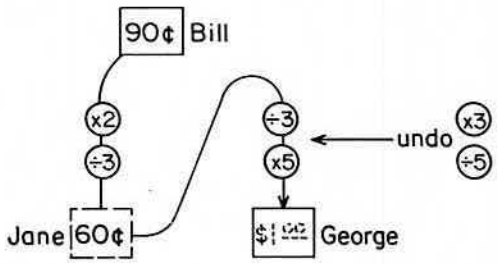
Example: Bill got an allowance of 90¢ a week. He spent $\frac{2}{3}$ of it. He paid $\frac{3}{5}$ of this amount for a used baseball.



Bill spent 36¢ for the baseball.

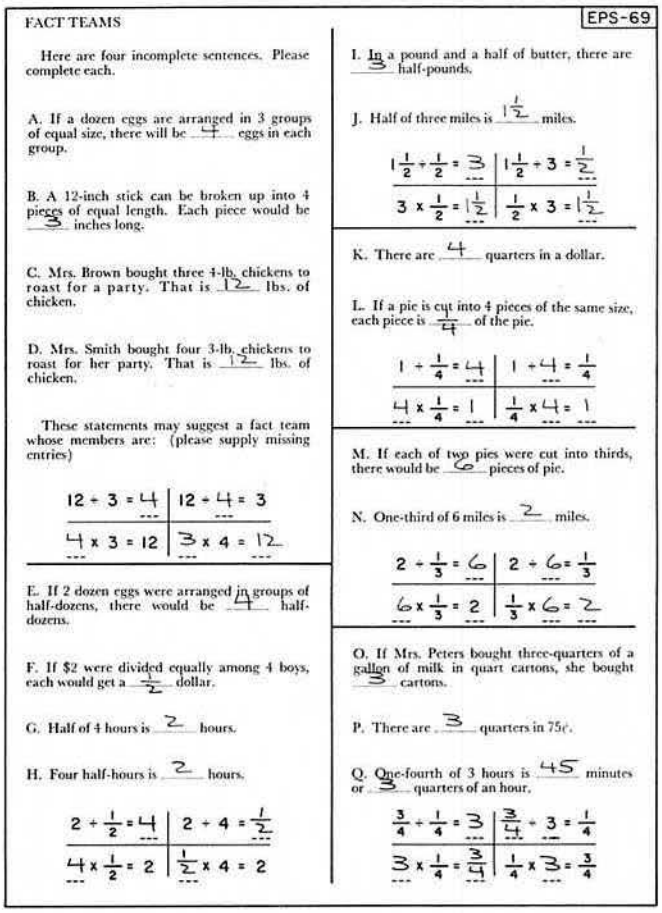
Division

Example: Bill got an allowance of 90¢. Jane got $\frac{2}{3}$ as much as Bill and $\frac{3}{5}$ as much as George. How much allowance did George receive?



Jane's 60¢ is $\frac{3}{5}$ of the amount George received. So we must undo the application of $\frac{3}{5}$, that is, undo the pair of operations ($\times 3$, $\div 5$). We can undo this pair by applying ($\div 3$, $\times 5$) to 60¢.

Practice sheet 69 introduces fact teams that capture the notions contained in everyday "fraction language."



FACT TEAMS — EPS-69

Here are four incomplete sentences. Please complete each.

A. If a dozen eggs are arranged in 3 groups of equal size, there will be 4 eggs in each group.

B. A 12-inch stick can be broken up into 4 pieces of equal length. Each piece would be 3 inches long.

C. Mrs. Brown bought three 4-lb. chickens to roast for a party. That is 12 lbs. of chicken.

D. Mrs. Smith bought four 3-lb. chickens to roast for her party. That is 12 lbs. of chicken.

These statements may suggest a fact team whose members are: (please supply missing entries)

$12 \div 3 = 4$ | $12 \div 4 = 3$

$4 \times 3 = 12$ | $3 \times 4 = 12$

E. If 2 dozen eggs were arranged in groups of half-dozens, there would be 4 half-dozens.

F. If $2 were divided equally among 4 boys, each would get a $\frac{1}{2}$ dollar.

G. Half of 4 hours is 2 hours.

H. Four half-hours is 2 hours.

$2 \div \frac{1}{2} = 4$ | $2 \div 4 = \frac{1}{2}$

$4 \times \frac{1}{2} = 2$ | $\frac{1}{2} \times 4 = 2$

I. In a pound and a half of butter, there are 3 half-pounds.

J. Half of three miles is $1\frac{1}{2}$ miles.

$1\frac{1}{2} \div \frac{1}{2} = 3$ | $1\frac{1}{2} \div 3 = \frac{1}{2}$

$3 \times \frac{1}{2} = 1\frac{1}{2}$ | $\frac{1}{2} \times 3 = 1\frac{1}{2}$

K. There are 4 quarters in a dollar.

L. If a pie is cut into 4 pieces of the same size, each piece is $\frac{1}{4}$ of the pie.

$1 \div \frac{1}{4} = 4$ | $1 \div 4 = \frac{1}{4}$

$4 \times \frac{1}{4} = 1$ | $\frac{1}{4} \times 4 = 1$

M. If each of two pies were cut into thirds, there would be 6 pieces of pie.

N. One-third of 6 miles is 2 miles.

$2 \div \frac{1}{3} = 6$ | $2 \div 6 = \frac{1}{3}$

$6 \times \frac{1}{3} = 2$ | $\frac{1}{3} \times 6 = 2$

O. If Mrs. Peters bought three-quarters of a gallon of milk in quart cartons, she bought 3 cartons.

P. There are 3 quarters in 75¢.

Q. One-fourth of 3 hours is 45 minutes or 3 quarters of an hour.

$\frac{3}{4} \div \frac{1}{4} = 3$ | $\frac{3}{4} \div 3 = \frac{1}{4}$

$3 \times \frac{1}{4} = \frac{3}{4}$ | $\frac{1}{4} \times 3 = \frac{3}{4}$

WHAT CAN YOU SAY?

1. Boyd had less than a quarter. He spent a fifth of his money for gum and three-fourths of his money for ice cream.

What can you say?

If Boyd had 12¢ then he spent 9 ¢ for ice cream.

If he had 15¢ then he spent 3 ¢ for gum.

Boyd spent LESS (more or less) for gum than he did for ice cream.

If Boyd spent 18¢ for ice cream then he had 24 ¢.

You can be sure that Boyd spent 4 ¢ for gum, 15 ¢ for ice cream, and had 1 ¢ left.

---

2. Two-ninths of the pupils of Edith's class were absent, and one-fourth of the pupils were at music practice. The class had less than 40 pupils.

What can you say?

If there were 27 pupils then 6 pupils would be absent. But there couldn't be 27 pupils, because "one-fourth of 27 pupils" wouldn't make much sense.

If there were 8 pupils at music practice then there must be 32 pupils in the class. But "two-ninths" of that number of pupils doesn't make much sense.

There must be an EVEN (odd or even) number of pupils absent.

If you think it through, you can be sure there are 8 pupils absent and 9 pupils at music practice . . . or a total of 17 pupils missing from the class and 19 pupils in the room.

---

3. One-sixth of the days this month are Sundays and one-sixth are Saturdays. The rest are school days.

What can you say?

There are 31 days in each of 7 months, 30 days in each of 4 months, and 28 days in February (except in Leap Years, when there are 29 days in February).

So, we can be sure the month referred to is either APRIL, JUNE, SEPT., or NOV.

There are 5 Sundays, 5 Saturdays, and 20 school days in the month.

We can be sure that the first day of the month was SATURDAY

---

4. One box was $\frac{3}{4}$ full of red marbles. Another box of the same size was $\frac{2}{3}$ full of blue marbles. The owner of the store filled the rest of both boxes with white marbles. He needed 7 white marbles.

What can you say?

The number of marbles in a full box must be divisible by both 3 and 4. Numbers less than 100 that can be divided evenly by both these numbers are 12, 24, 36, 48, 60, 72, 84, and 96

If there were 24 in each box then there would be 18 red and 16 blue marbles. The owner would need 14 white marbles to fill both boxes.

There must be 12 in each full box. The owner needed 3 white ones to fill the box with the red marbles. He needed 4 white ones to fill the box with the blue marbles.

Notes on Pupil Page 70

Tasks: *Pupils read the* What Can You Say? *situations and fill in the missing entries.*

Purpose: To provide open-ended questions which require the use of fractions.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: A good mathematical problem may have no pat answer. Rather, it presents a situation and poses the question: think about the situation; what can you say?

Example: Mabel collected exactly twice as many used magazines as June and three times as many as George. What can you say?

Not very much information, but what can you do with it?

One approach would be to make a series of statements beginning, "If Mabel collected 12 (or some other number of) used magazines then June had collected. . . ."

This might lead to an interesting and fruitful discussion—probing the relationships among the data given. Suppose that we made records of some "If . . . then . . ." statements. We very soon recognize that the number of magazines Mabel has must be divisible by 6. As we think about it, we realize that we know that the number June has must be divisible by 3 and that George must have collected an even number of magazines.

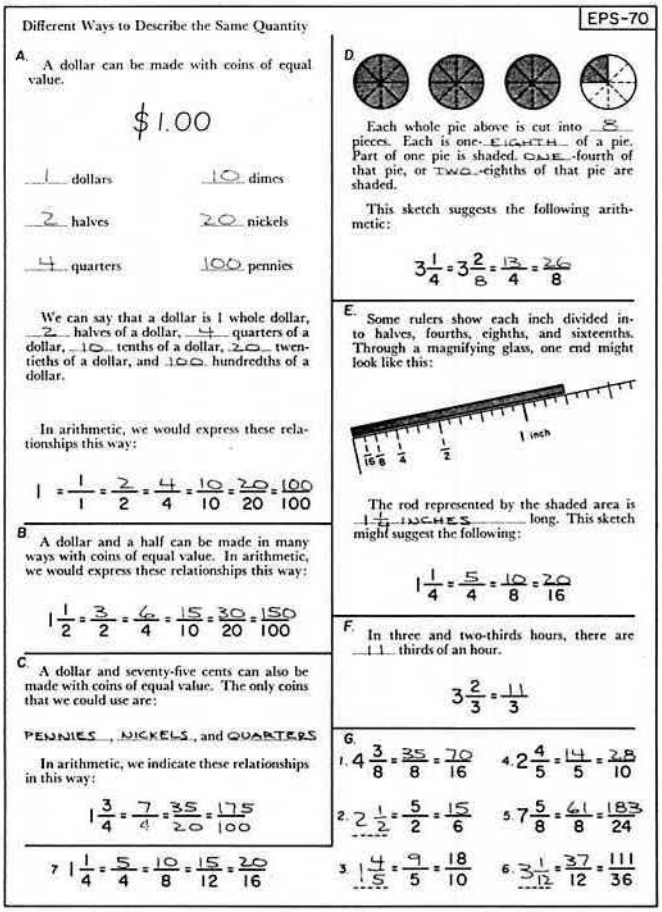
If Mabel has	6	12	18	24	30	
then June has	3	6	9	12	15	
and George has	2	4	6	8	10	
Total	11	22	33	44	55	

And, as we note the total, we can say something else: the total number of magazines the three have collected is a multiple of 11. We might also point out that June has 1½ times as many as George.

That's quite a lot to say—and it seemed that we started with so little! This is mathematics, and it's within the reach of pupils who are encouraged to think!

Encourage pupils to think about each situation on page 70. As they supply the missing information, they may notice surprising relationships. For example, notice that, in the first example, Boyd must have a multiple of 20¢ that is less than a quarter—so he must have exactly 20¢.

Practice sheet 70 looks at many ways to talk about different amounts of money, about 3¾ pies, 1¼ inches, 3⅔ hours, and summarizes the discussion with appropriate arithmetic statements. This is actually an exercise in finding notation that captures the ideas behind familiar expressions.



Different Ways to Describe the Same Quantity

EPS-70

A. A dollar can be made with coins of equal value.

$1.00

1 dollars — 10 dimes

2 halves — 20 nickels

4 quarters — 100 pennies

We can say that a dollar is 1 whole dollar, 2 halves of a dollar, 4 quarters of a dollar, 10 tenths of a dollar, 20 twentieths of a dollar, and 100 hundredths of a dollar.

In arithmetic, we would express these relationships this way:

$1 = \frac{1}{1} = \frac{2}{2} = \frac{4}{4} = \frac{10}{10} = \frac{20}{20} = \frac{100}{100}$

B. A dollar and a half can be made in many ways with coins of equal value. In arithmetic, we would express these relationships this way:

$1\frac{1}{2} = \frac{3}{2} = \frac{6}{4} = \frac{15}{10} = \frac{30}{20} = \frac{150}{100}$

C. A dollar and seventy-five cents can also be made with coins of equal value. The only coins that we could use are:

PENNIES, NICKELS, and QUARTERS

In arithmetic, we indicate these relationships in this way:

$1\frac{3}{4} = \frac{7}{4} = \frac{35}{20} = \frac{175}{100}$

7. $1\frac{1}{4} = \frac{5}{4} = \frac{10}{8} = \frac{15}{12} = \frac{20}{16}$

D.

Each whole pie above is cut into 8 pieces. Each is one-EIGHTH of a pie. Part of one pie is shaded. ONE-fourth of that pie, or TWO-eighths of that pie are shaded.

This sketch suggests the following arithmetic:

$3\frac{1}{4} = 3\frac{2}{8} = \frac{13}{4} = \frac{26}{8}$

E. Some rulers show each inch divided into halves, fourths, eighths, and sixteenths. Through a magnifying glass, one end might look like this:

1 inch

The rod represented by the shaded area is 1¼ INCHES long. This sketch might suggest the following:

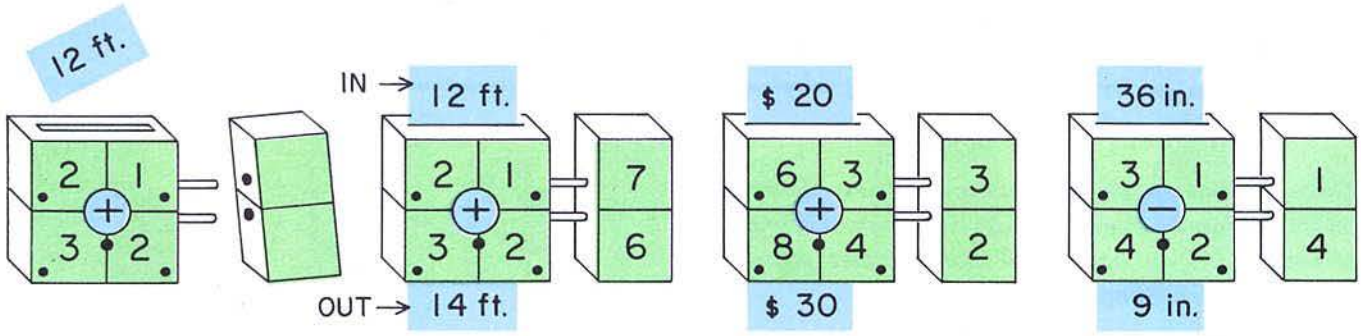
$1\frac{1}{4} = \frac{5}{4} = \frac{10}{8} = \frac{20}{16}$

F. In three and two-thirds hours, there are 11 thirds of an hour.

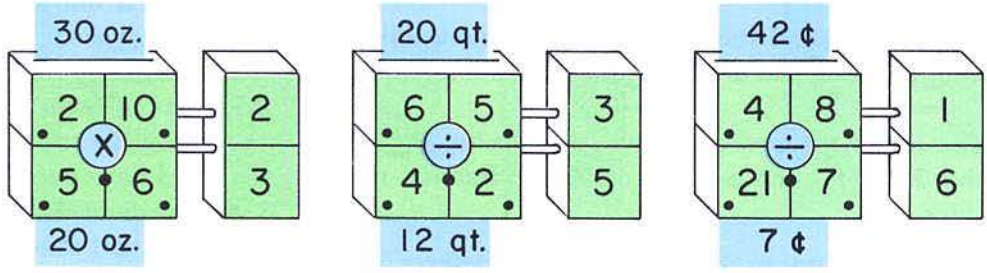
$3\frac{2}{3} = \frac{11}{3}$

G.

1. $4\frac{3}{8} = \frac{35}{8} = \frac{70}{16}$
2. $2\frac{1}{2} = \frac{5}{2} = \frac{15}{6}$
3. $1\frac{4}{5} = \frac{9}{5} = \frac{18}{10}$
4. $2\frac{4}{5} = \frac{14}{5} = \frac{28}{10}$
5. $7\frac{5}{8} = \frac{61}{8} = \frac{183}{24}$
6. $3\frac{1}{12} = \frac{37}{12} = \frac{111}{36}$



Mr. Wilson finds a SIMPLIFIER for Alec's FRACTION MACHINE.



Have you any hunches about the Simplifier? What is its purpose?

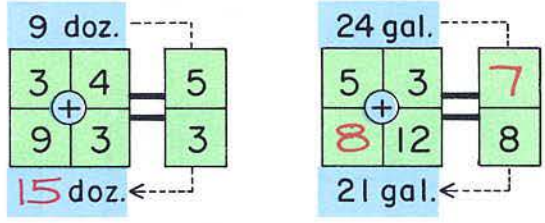
"I thought those two pins on the side of Alec's machine were handles," Mr. Wilson admitted. "Then I found the 'Simplifier.'

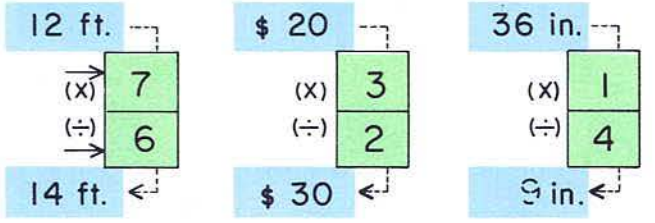
"As soon as I plugged the Simplifier onto those two pins, numbers appeared in its two windows. Examples of results I found are given above.

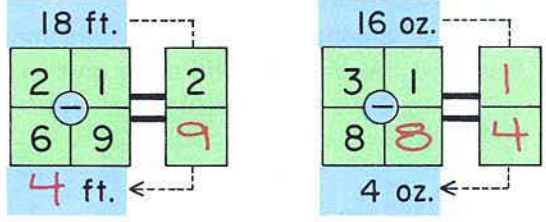
"I began to see what Alec had in mind when I kept notes on just the inputs, outputs, and the Simplifier.

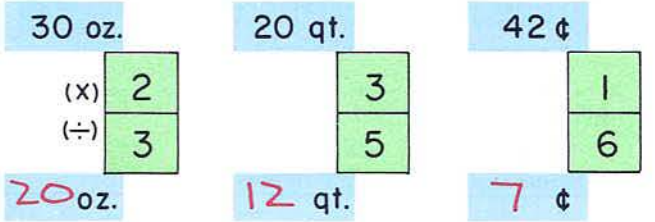
"As I see it, the Simplifier shows a shortcut from the input to the output (as if it were very much like the simple fraction machine I experimented with first).

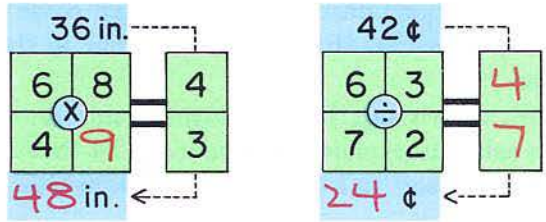
"Here are more of my records of experiments." *(Please fill in the missing entries.)*











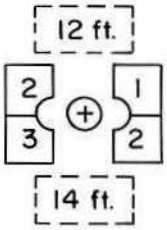
71-72 Simplifier machines to replace Alec's fraction machine (Basic operations of +. -, x, and / of fractions explored)

Tasks: *Pupils study the examples, read the story, and fill in the missing entries.*

Purpose: To lay the groundwork for the notion that an indicated sum, difference, product, or quotient of fractions can be replaced by a single fraction.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: All investigations thus far with Alec's machines have involved two pairs of operations and a sign (+, −, ×, or ÷).



When we select an input such as 12 feet, the machine carries out a sequence of operations. We know enough about the machine to predict the output.

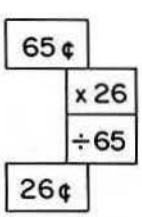
12 ft. ÷ 3 = 4 ft.; 4 ft. × 2 = 8 ft.
12 ft. ÷ 2 = 6 ft.; 6 ft. × 1 = 6 ft.
8 ft. + 6 ft. = 14 ft.

Alec's Simplifier, in effect, points out the fact that there is a much simpler way to get from 12 ft. to 14 ft.:

12 ft. ÷ 6 = 2 ft.; and 2 ft. × 7 = 14 ft.

Is there such a shortcut for all allowable settings of Alec's machines?

Urge pupils to discuss this question, giving reasons for their speculations. Select some examples from earlier pages. Try to find a shortcut around each. There is, of course, a pair of operations that will lead from every input thus far considered to the same output Alec's machine would produce:



or

65 (×26) (÷65) 26

65 (×65) (÷26) 26

True, this simple-minded approach may not give us the pair of operations which used the smallest numbers:

65 (×2) (÷5) 26

but it does convince us that we can always find a single pair of operations leading from input to output. We hope that you will not point this out to the pupils, at least for a few more pages.

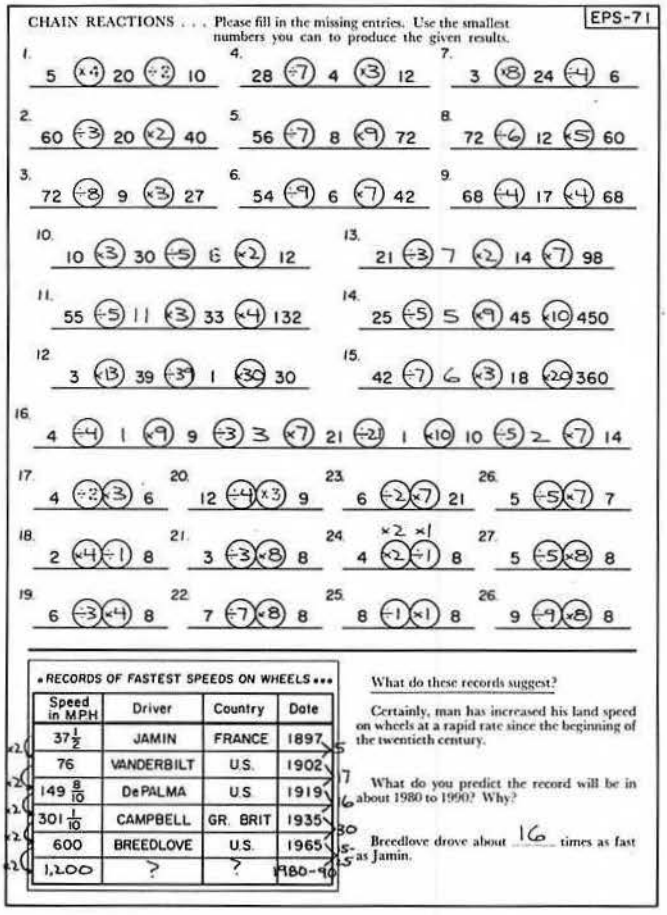
Practice sheet 71 asks the pupils to begin supplying their own operations (and intermediate outputs) that will connect an input to an output:

6 ◯ ◯ 8

They could find as many pairs as they like, such as:

(×4) (÷3) (×8) (÷6) (×12) (÷9) (×16) (÷12)

and their "reverses," but the instruction to use the smallest possible numbers narrows the choice to the first pair and its reverse.


CHAIN REACTIONS . . . Please fill in the missing entries. Use the smallest numbers you can to produce the given results. EPS-71

1. 5 (×4) 20 (÷2) 10
2. 60 (÷3) 20 (×2) 40
3. 72 (÷8) 9 (×3) 27
4. 28 (÷7) 4 (×3) 12
5. 56 (÷7) 8 (×9) 72
6. 54 (÷9) 6 (×7) 42
7. 3 (×8) 24 (÷4) 6
8. 72 (÷6) 12 (×5) 60
9. 68 (÷4) 17 (×4) 68
10. 10 (×3) 30 (÷5) 6 (×2) 12
11. 55 (÷5) 11 (×3) 33 (×4) 132
12. 3 (×13) 39 (÷3) 1 (×30) 30
13. 21 (÷3) 7 (×2) 14 (×7) 98
14. 25 (÷5) 5 (×9) 45 (×10) 450
15. 42 (÷7) 6 (×3) 18 (×20) 360
16. 4 (÷4) 1 (×9) 9 (÷3) 3 (×7) 21 (÷21) 1 (×10) 10 (÷5) 2 (×7) 14
17. 4 (÷2) (×3) 6
18. 2 (×4) (÷1) 8
19. 6 (÷3) (×4) 8
20. 12 (÷4) (×3) 9
21. 3 (÷3) (×8) 8
22. 7 (÷7) (×8) 8
23. 6 (÷2) (×7) 21
24. 4 (×2 ×1) (×2) (÷1) 8
25. 8 (÷1) (×1) 8
26. 5 (÷5) (×7) 7
26. 5 (÷5) (×8) 8
26. 9 (÷9) (×8) 8

• RECORDS OF FASTEST SPEEDS ON WHEELS •••

Speed in MPH	Driver	Country	Date
37½	JAMIN	FRANCE	1897
76	VANDERBILT	U.S.	1902
149 8/10	DePALMA	U.S.	1919
301 1/10	CAMPBELL	GR. BRIT	1935
600	BREEDLOVE	U.S.	1965
1,200	?	?	1980-90

What do these records suggest?

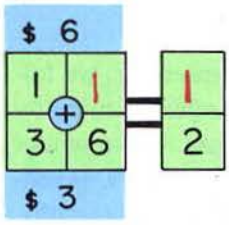
Certainly, man has increased his land speed on wheels at a rapid rate since the beginning of the twentieth century.

What do you predict the record will be in about 1980 to 1990? Why?

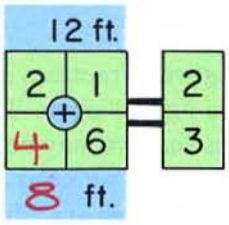
Breedlove drove about 16 times as fast as Jamin.

Use every shortcut you can to find the missing entries. Be sure to check your work.

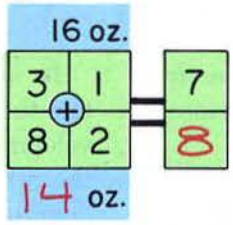
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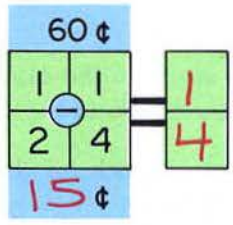
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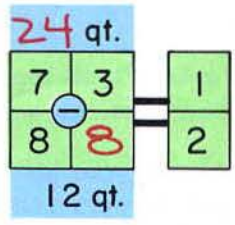
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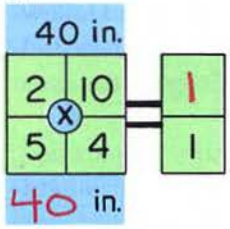
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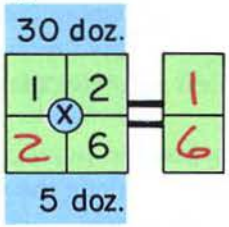
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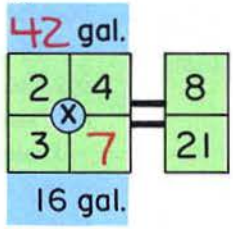
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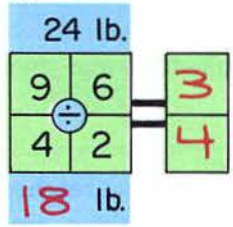
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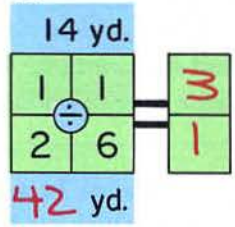
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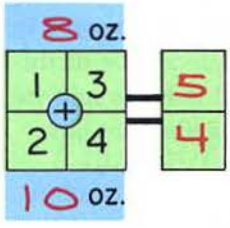
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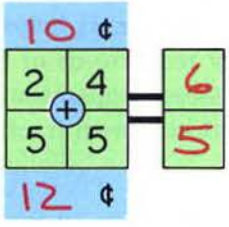
Use any amounts you wish as inputs.

+ MANY POSSIBLE INPUTS.

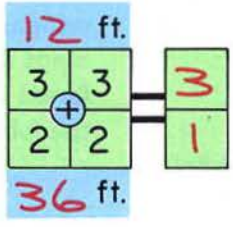
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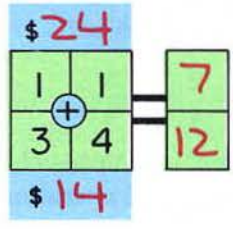
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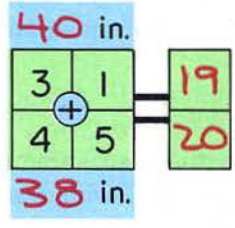
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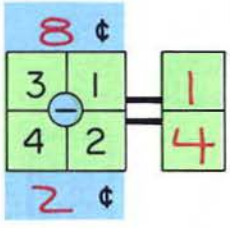
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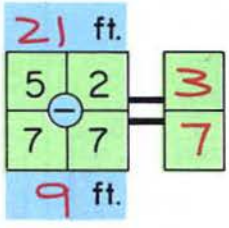
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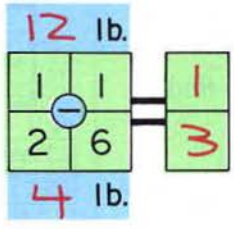
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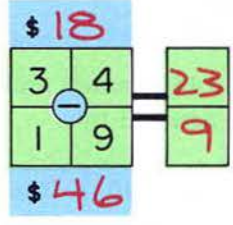
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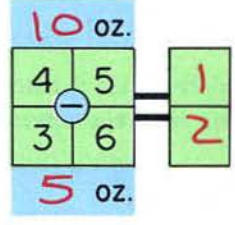
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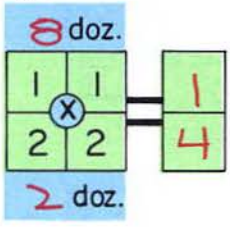
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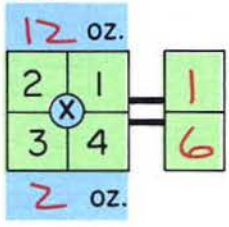
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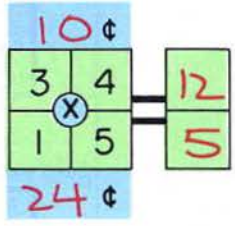
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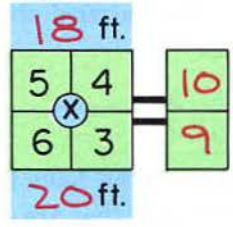
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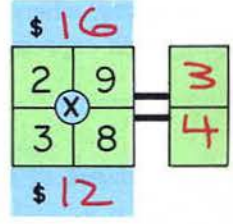
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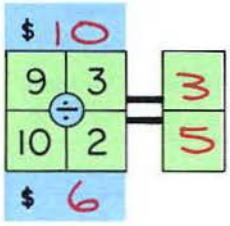
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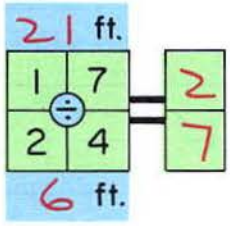
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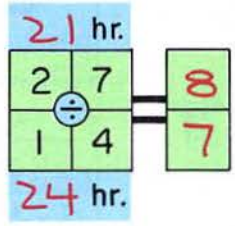
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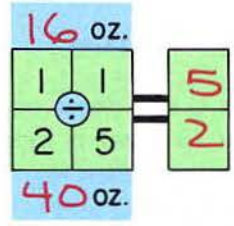
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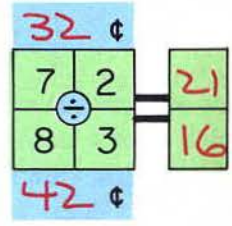
28.



29.



30.



Check back over the examples above. Are the Simplifier settings the smallest possible settings that would lead to the correct results? Alec's Simplifier always chooses the smallest possible settings.

Tasks: *Pupils supply the missing entries, choosing their own inputs in exercises* 11-30. *They also check to see that they have used the smallest possible settings on the Simplifier.*

Purpose: To lay the groundwork for computing with fractions and reducing fractions to lowest terms.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Suppose that we consider "$\frac{1}{3}+\frac{1}{6}$" as shorthand for the settings on Alec's fraction machine.

a. Decide on an input amount of whatever you like;

b. divide the amount by 3 and multiply the result by 1;

c. similarly divide the original amount by 6 and multiply the result by 1;

d. find the sum of the results in *b* and *c*.

Here is the shorthand for the setting on Alec's Simplifier:

a. Decide on an input amount of whatever you like;

b. divide that amount by 2 and multiply the result by 1.

Now suppose that we select the amount 12 eggs. It will turn out that

A: $\frac{1}{3}+\frac{1}{6}$ or B: $\frac{1}{2}$

the final result is 6 eggs. Which machine was used? You have no way of knowing!

When other amounts are selected and machine A or B is used, the outcome will never give us a clue as to which machine was used. We can capture that idea by writing:

$$\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$$

By a rather unusual route we have come upon a result that is both usual and familiar to the initiated.

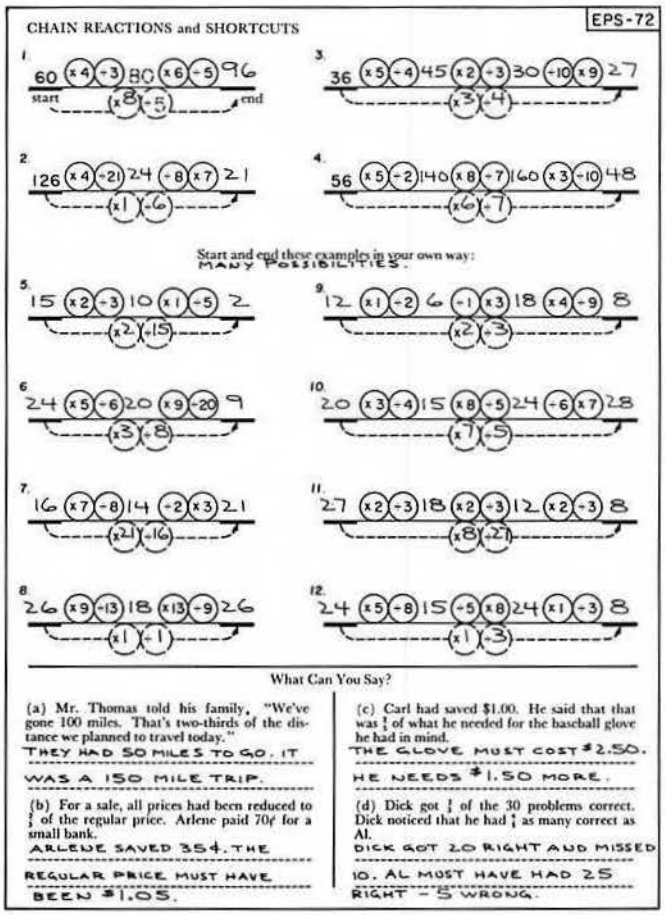
Yet, if, without having traveled this route, we were asked to explain what we mean by that result, we might resort to a pie diagram such as this:



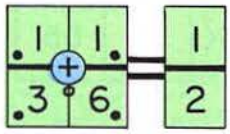
Our explanation: (a) start with a pie, or any amount; (b) divide it into three parts of the same size and shade one of them ($\div 3$, $\times 1$); (c) divide a new pie of the same size into six parts of the same size and shade one of them ($\div 6$, $\times 1$); (d) put the shaded pieces together to represent the sum of (b) and (c). We could have ended with the same amount of pie if we had divided the given pie in two parts of the same size, and shaded one of the parts.

But these are precisely the steps we took with the machines. If the route we took was unfamiliar then it will have developed a deeper understanding of the fundamental ideas involved.

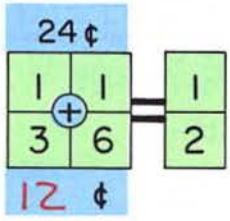
Practice sheet 72 prepares for the same kind of illumination of multiplication and division of fractions. (See pupil pages 81 and 82.)

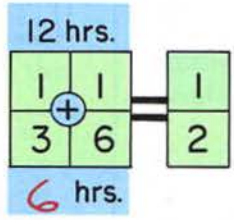


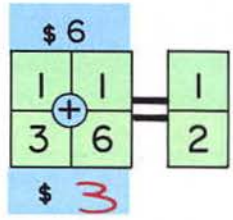
"One day, during my experiments, it dawned on me that as soon as the 5 selector knobs on Alec's machine were set, the Simplifier was automatically set before I had selected an input. Also, as I changed inputs without changing settings, the Simplifier remained unchanged."

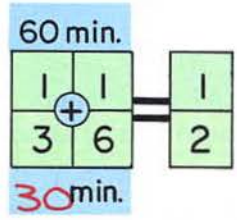


$\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

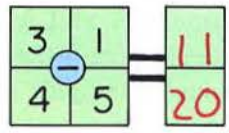








"Think of anything! Is $\frac{1}{3}$ of it plus $\frac{1}{6}$ of it equal to $\frac{1}{2}$ of it?"



A third of Bill's marbles are red. A sixth of them are blue. The rest are brown. What can you say?

If he has any marbles at all, the fewest he can have is 6 marbles. In that event, he would have 2 red and 1 blue marble(s).

If he had more than 20 but less than 30 marbles then he would have to have 8 red marbles and 4 blue marbles.

Can you be sure that one-half of the marbles are brown? YES (Yes or no)

Can you say that Bill has twice as many red marbles as blue marbles? YES (Yes or no)

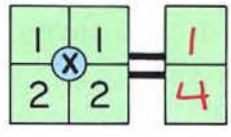
If Bill has 7 blue marbles then he has a total of 42 marbles.

Bill put his red and brown marbles together. There were 30. Bill had 6 blue marbles.

Mary and Jane collected stamps. They had the same number of stamps. Three-fourths of Mary's stamps were U. S. stamps. One-fifth of Jane's stamps were U. S. stamps. What can you say?

The fewest stamps each could have had is 20 stamps.

If that were the case, Mary had 15 U. S. stamps and Jane had 4 U. S. stamps. Mary had 11 more U. S. stamps than Jane had.



A half of a half-dollar is a QUARTER.

Allen put half of his allowance in the bank. He spent half of the rest for a radio tube.

What can you say? HE SPENT $\frac{1}{4}$ OF HIS ALLOWANCE FOR A TUBE.

73 Steps toward generalizing these specific investigations

Tasks: *Pupils read the story at the top, fill in the missing entries, and fill the blanks in the discussions.*

Purpose: To lay the groundwork for dealing with abstract statements such as "$\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$," "$\frac{3}{4}-\frac{1}{5}=1\frac{1}{20}$," and "$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$."

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: In a sense, we have reached the point toward which we have been working—we have separated these pairs of operations from eggs, inches, and dollars and have begun to treat fractions as abstractions, as having an existence of their own. We shall make statements such as:

$$\frac{1}{2}+\frac{1}{4}=\frac{2}{4}+\frac{1}{4}=\frac{3}{4}$$

without selecting an input amount such as 60 minutes, following the sequence of operations denoted by:

$$\frac{1}{2}+\frac{1}{4}$$

to find that it leads us to 45 minutes, and then finding that the operations denoted by:

$$\frac{3}{4}$$

would have produced the same result.

However, whenever we are puzzled about simplifying such expressions as:

$$\frac{5}{9}+\frac{3}{4}$$

we can resort to the methods we have been using. Further, if we make statements that are questioned, such as:

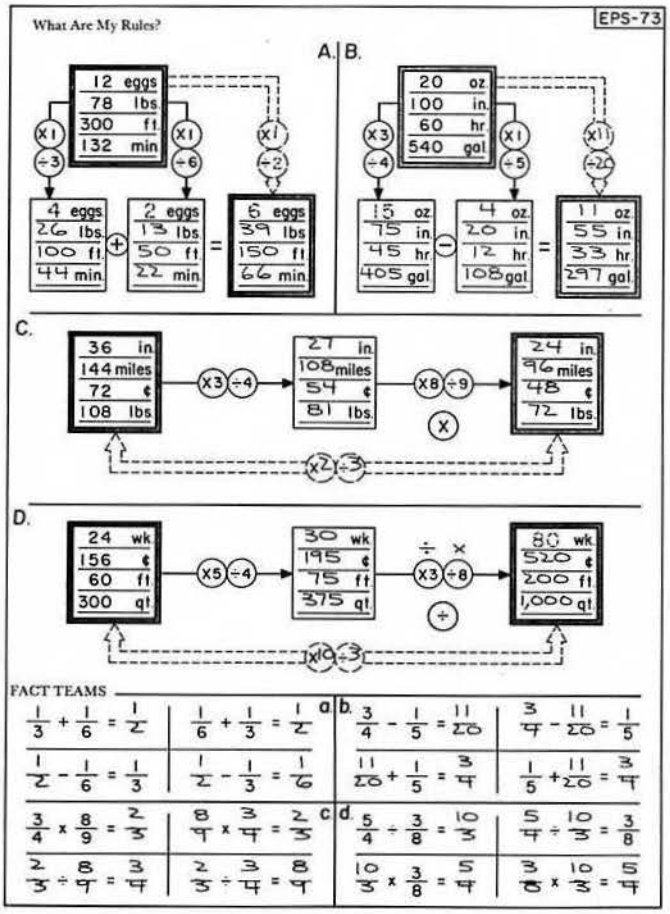
$$\frac{1}{4}+\frac{2}{3}=\frac{3}{7}\ \text{(true or false?)}$$

we can find out whether ¼ of a dozen eggs (3 eggs) plus ⅔ of a dozen eggs (8 eggs) is indeed ⅜ of a dozen eggs. Our test tells us that $1\frac{1}{12}$ (3 eggs + 8 eggs is 11 eggs) is the sum of ¼ and ⅔. Is $1\frac{1}{12}$ the same as ⅜? Is $1\frac{1}{12}$ of 84 eggs the same as ⅜ of 84 eggs? No.

In working with fractions as abstractions, we follow the rules agreed upon. The danger in such developments is that we may not be sure we remember the rules. However, we need not feel utterly dependent on our memories in adding, subtracting, multiplying, and dividing fractions, because we can go back to selecting an amount of something and following the sequence of operations given—and then finding the simplest sequence that will produce the same end result.

Unless we provide pupils with this preparation, many will flounder along the way among a proliferation of rules for this and that. Many will not even know when they are lost, let alone have any notions about how to find their way back to familiar ground.

Practice sheet 73 presents a summary of the developments thus far. It helps show that changes in input simply reassure us of the equivalence of different sequences of operations, and it helps initiate the job of working with fractions as abstractions.



X Addition and Subtraction of Fractions

Is it true that, for all numbers a, b, and c such that $c \neq 0$ and both a and b are multiples of c,

$$(a \div c) + (b \div c) = (a + b) \div c \ ?$$

What does the question mean?

Lists of 3 such numbers are given in the chart below. Follow the clues to supply the missing entries.

	I	II	III	IV	V	VI
a	6	20	15	60	63	40
b	8	35	15	40	28	56
c	2	5	3	10	7	8
a÷c	3	4	5	6	9	5
b÷c	4	7	5	4	4	7
(a÷c)+(b÷c)	7	11	10	10	13	12
a+b	14	55	30	100	91	96
(a+b)÷c	7	11	10	10	13	12

In each column of the chart, is it true that

$$(a \div c) + (b \div c) = (a + b) \div c \ ?$$

Notice that each group of three numbers (a, b, and c) was selected so that both a and b could be evenly divided by c. This is because, at this point, we wish to avoid fractions. Why did we say that c should not be 0?

Here's another way to arrange our investigation:

a	b	c	(a÷c)+(b÷c)	(a+b)÷c
8	12	4	2 + 3	20 ÷ 4
15	21	3	5 + 7	36 ÷ 3
28	14	7	4 + 2	42 ÷ 7
18	12	3	6 + 4	30 ÷ 3
45	36	9	5 + 4	81 ÷ 9
72	24	8	9 + 3	96 ÷ 8

Make your own selections of three numbers (a, b, and c). Remember to choose them to avoid fractions and to avoid dividing by 0.

	VII	+ VIII	+ IX	+ X	+ XI	+ XII
a	42	30	15	27	24	72
b	28	20	9	9	18	36
c	7	5	3	9	6	12
a÷c	6	6	5	3	4	6
b÷c	4	4	3	1	3	3
(a÷c)+(b÷c)	10	10	8	4	7	9
a+b	70	50	24	36	42	108
(a+b)÷c	10	10	8	4	7	9

What is your conclusion?

Is it true that, for all a, b, and c such that $c \neq 0$, a and b are multiples of c, and $a > b$,

$$(a \div c) - (b \div c) = (a - b) \div c \ ?$$

Here is an incomplete record of an experiment:

a	b	c	(a÷c)−(b÷c)	(a−b)÷c
30	12	3	10 − 4	18 ÷ 3
40	15	5	8 − 3	25 ÷ 5
60	28	4	15 − 7	32 ÷ 4

An incomplete record of another experiment:

			+	+	+
a	42	63	56	100	35
b	24	36	21	50	25
c	6	9	7	2	5
a÷c	7	7	8	50	7
b÷c	4	4	3	25	5
(a÷c)−(b÷c)	3	3	5	25	2
a−b	18	27	35	50	10
(a−b)÷c	3	3	5	25	2

What is your conclusion?

+ MANY ANSWERS POSSIBLE.

74-76 Exploring the usual algorithm using common denominators

Tasks: *Pupils complete the tables.*

Purpose: To verify certain generalizations for adding or subtracting quotients.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Pupil pages 74-76 are designed to establish patterns for adding and subtracting fractions. By using quantities and regarding fractions as pairs of operations to be applied to quantities, we can verify through examples (as the pupil has already done on preceding pages) that, for all numbers a, b, and c such that $c \neq 0$ and $a \geq b$,

$$(1)\quad \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

and

$$(2)\quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

We can also verify something about quotients through examples (as the pupil is asked to do on pupil page 74)—namely that, for all numbers a, b, and c such that a and b are multiples of c, $c \neq 0$, and $a \geq b$,

$$(3)\quad (a \div c) + (b \div c) = (a+b) \div c$$

and

$$(4)\quad (a \div c) - (b \div c) = (a-b) \div c.$$

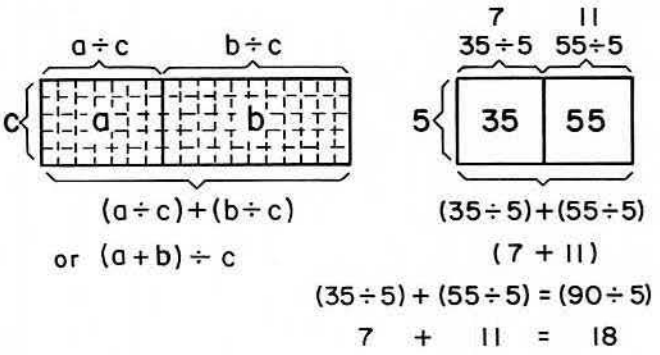
Now, notice that applying a quotient $(a \div c)$ to a quantity has the same effect as applying the fraction a/c (or $\times a$, $\div c$) to the quantity. For example,

$(10 \div 5)$ of 15 cents = 15 cents $\times 2$ = 30 cents

$\frac{10}{5}$ of 15 cents = (15 cents $\times 10) \div 5$ = 30 cents

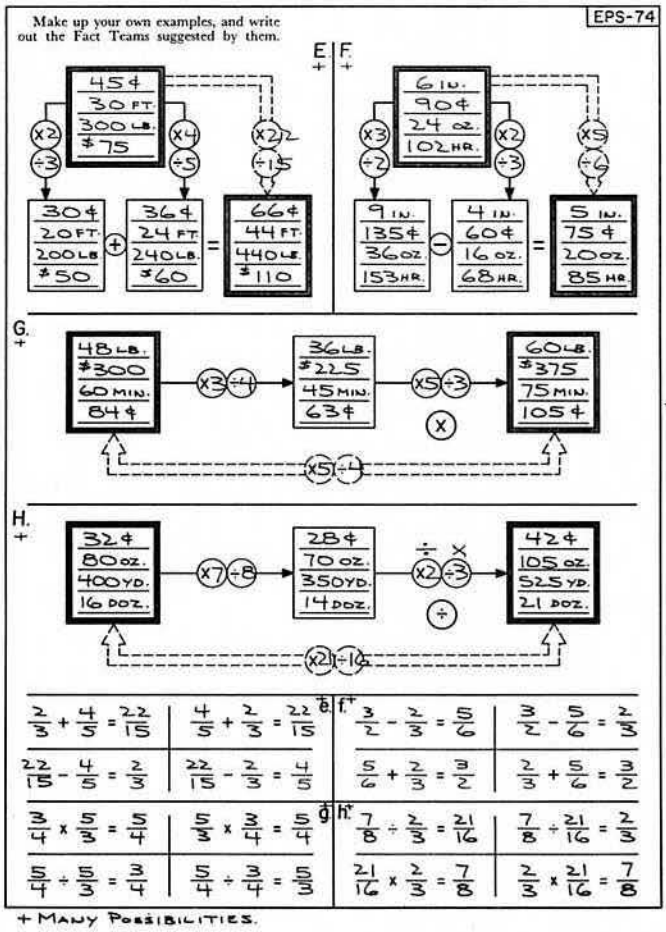
This equivalence leads us to agree that quotients may be treated as fractions. Thus, patterns 3 and 4 are merely special cases of patterns 1 and 2, respectively.

Pupils may gain further insight into patterns 3 and 4 by looking at a sketch such as the following:



Sketches like this are not helpful in checking patterns 1 and 2 in cases where a or b is not a multiple of c. In such cases, the use of "input" quantities (which are multiples of c units) will do the job.

Practice sheet 74 gives the pupils an opportunity to make up examples of their own in the style of practice sheet 73. This will help them grasp more completely the four basic ways of combining pairs of operations.



More experiments may convince you that, for all whole numbers a, b, and c such that a and b are multiples of c, $c \neq 0$, and $a \geq b$,

$$(a \div c) + (b \div c) = (a + b) \div c$$

and

$$(a \div c) - (b \div c) = (a - b) \div c .$$

Some mathematicians refer to these as patterns for the distributive principles for division over addition and over subtraction.

You have already used these principles many times — when you had such problems as $42 \div 3 = ?$ You might have thought about it this way:

$$3\overline{)42} = 3\overline{)30 + 12}$$
$$10 + 4 = 14$$

You could have recorded your computations in this way:

$$(30 \div 3) + (12 \div 3) = (30 + 12) \div 3$$

In arithmetic, when we write one number above and another below a bar, we agree that we mean to divide the top number (numerator) by the bottom number (denominator). We could write $12 \div 3$ in this way:

$$\frac{12}{3} \quad \begin{matrix} \leftarrow \text{Numerator} \\ \leftarrow \text{Denominator} \end{matrix}$$

Alec would use other names:

$$\frac{12}{3} \quad \begin{matrix} \leftarrow \text{Multiplier (machine)} \\ \leftarrow \text{Divider (machine)} \end{matrix}$$

So we could write :

$$\frac{30}{3} + \frac{12}{3} = \frac{30 + 12}{3} = \frac{42}{3}$$

and we could rewrite the patterns for the distributive principles:

$$(a \div c) + (b \div c) = (a + b) \div c$$
$$(a \div c) - (b \div c) = (a - b) \div c$$

in this way:

I $\quad \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ $\qquad$ II $\quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$

Let's experiment further.

			I $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$	II $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$
1.	a	30	$\frac{30}{3}+\frac{12}{3}=\frac{42}{3}$	$\frac{30}{3}-\frac{12}{3}=\frac{18}{3}$
	b	12		
	c	3	$10 + 4 = 14$	$10 - 4 = 6$
2.	a	40	$\frac{40}{5}+\frac{30}{5}=\frac{70}{5}$	$\frac{40}{5}-\frac{30}{5}=\frac{10}{5}$
	b	30		
	c	5	$8 + 6 = 14$	$8 - 6 = 2$
3.	a	63	$\frac{63}{7}+\frac{21}{7}=\frac{84}{7}$	$\frac{63}{7}-\frac{21}{7}=\frac{42}{7}$
	b	21		
	c	7	$9 + 3 = 12$	$9 - 3 = 6$
4.	a	88	$\frac{88}{8}+\frac{56}{8}=\frac{144}{8}$	$\frac{88}{8}-\frac{56}{8}=\frac{32}{8}$
	b	56		
	c	8	$11 + 7 = 18$	$11 - 7 = 4$

More experiments.

	I	II
5.	$\frac{12}{2}+\frac{4}{2}=\frac{16}{2}=8$	$\frac{12}{2}-\frac{4}{2}=\frac{8}{2}=4$
6.	$\frac{45}{5}+\frac{15}{5}=\frac{60}{5}=12$	$\frac{45}{5}-\frac{15}{5}=\frac{30}{5}=6$
7.	$\frac{42}{6}+\frac{18}{6}=\frac{60}{6}=10$	$\frac{42}{6}-\frac{18}{6}=\frac{24}{6}=4$
8.	$\frac{14}{4}+\frac{6}{4}=\frac{20}{4}=5$	$\frac{14}{4}-\frac{6}{4}=\frac{8}{4}=2$
9.	$\frac{33}{6}+\frac{15}{6}=\frac{48}{6}=8$	$\frac{33}{6}-\frac{15}{6}=\frac{18}{6}=3$
10.	$\frac{17}{2}+\frac{7}{2}=\frac{24}{2}=12$	$\frac{17}{2}-\frac{7}{2}=\frac{10}{2}=5$

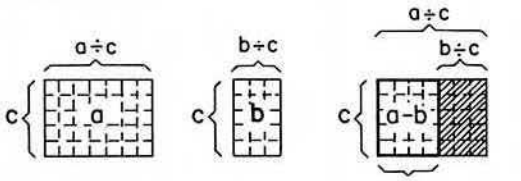
Do you notice a difference between examples above the heavy line and below it?

Tasks: *Pupils complete the tables.*

Purpose: To verify generalizations for adding or subtracting quotients and fractions.

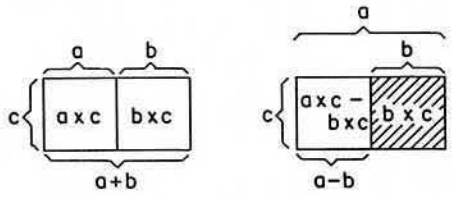
Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: A sketch may help strengthen the understanding of the distributive principle for division over subtraction:



$(a \div c) - (b \div c)$ or $(a - b) \div c$

Perhaps an additional support can be found in recalling the distributive principles for multiplication over addition and over subtraction. Here are sketches and the patterns:



$(a \times c) + (b \times c) = (a + b) \times c$ $\quad$ $(a \times c) - (b \times c) = (a - b) \times c$

The distributive principle for multiplication over addition is, as you will see quickly, the basic principle behind the usual multiplication algorithm. For example, to multiply 17 by 5 we partition 17, multiply each part by 5, and add the partial products:

$$\begin{aligned} 17 \times 5 = (10 + 7) \times 5 &= 10 \times 5 + 7 \times 5 \\ &= 50 + 35 \\ &= 85 \end{aligned}$$

It may be well to write down the pattern for the four variations of this useful idea:

$$(a \times c) + (b \times c) = (a + b) \times c$$
$$(a \times c) - (b \times c) = (a - b) \times c$$
$$(a \div c) + (b \div c) = (a + b) \div c$$
$$(a \div c) - (b \div c) = (a - b) \div c$$

$$\boxed{(a \overset{\times}{\div} c) \pm (b \overset{\times}{\div} c) = (a \pm b) \overset{\times}{\div} c}$$

Encourage pupils to test these patterns.

There is less difference than one might think between the usual language for fractions and Alec's language.

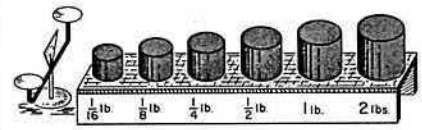
Numerator ⟶ 2 ⟵ Multiplier Machine
Denominator ⟶ 5 ⟵ Divider Machine

When one applies a fraction, say $\frac{2}{5}$, to a quantity, he may first divide the quantity by 5 to obtain a new unit or *denomination;* next, he counts out or *enumerates* 2 of the new units. Thus, 5 is the *denominator* and 2 is the *numerator.*

Practice sheet 75 deals with an application of fractions to weights. (The table on the right reveals the familiar pattern of the binary system. This pattern will always appear in any system in which each unit is twice the size of the next smaller unit, as in gallons, half-gallons, quarts, pints, and half-pints.)

EPS-75

Jim had a part-time job in a chemical laboratory. One of his jobs was to package various chemicals. Orders always came to him for ounces of this or that chemical.



He had just one of each of the weights shown above to use with his chemical balance.

He knew that there are 16 ounces in one pound.

Some orders he received are listed below. What weights should he use for each order? Check the weights used as in example A.

	ORDERS	Weights — in pounds 2	1	1/2	1/4	1/8	1/16
A	6 ounces of sodium (Na)				✓	✓	
B	3 ounces of copper (Cu)					✓	✓
C	8 ounces of sulphur (S)			✓			
D	7 ounces of silver (Ag)				✓	✓	✓
E	11 ounces of chlorine (Cl)			✓		✓	✓
F	1 pound of zinc (Zn)		✓				
G	1½ pounds of aluminum (Al)		✓	✓			
H	15 ounces of mercury (Hg)			✓	✓	✓	✓
I	¼ pound of carbon (C)			✓			
J	¾ pound of iron (Fe)			✓	✓		
K	27 ounces of calcium (Ca)		✓	✓		✓	✓
L	3 lb., 9 oz. of lead (Pb)	✓	✓	✓			✓

Jim made a time saver chart like the one on the right. Please complete the chart.

Weights in pounds 2	1	1/2	1/4	1/8	1/16	oz.
					•	1
				•	•	2
			•			3
			•		•	4
			•	•		5
			•	•	•	6
		•				7
		•			•	8
		•		•		9
		•		•	•	10
		•	•			11
		•	•		•	12
		•	•	•		13
		•	•	•	•	14
	•					15
	•				•	16
	•			•		17
	•			•	•	18
	•		•			19
	•		•		•	20
	•		•	•		21
	•		•	•	•	22
	•	•				23
	•	•			•	24
	•	•		•		25
	•	•		•	•	26
	•	•	•			27
	•	•	•		•	28
	•	•	•	•		29
	•	•	•	•	•	30
•						31
•					•	32
•				•		33
•				•	•	34
•			•			35
•			•		•	36
•			•	•		37
•			•	•	•	38
•		•				39

$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ and $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$

Let's check the results in examples similar to the last ones on the previous page.

$\frac{6}{4} + \frac{2}{4} = \frac{8}{4} = 2$	$\frac{6}{4} - \frac{2}{4} = \frac{4}{4} = 1$

Suppose we are talking about gallons of milk. We know that a fourth or quarter of a gallon is called a "quart."

Mrs. Jones had 6 quarts of milk and bought 2 more quarts of milk. Together she had 8 quarts of milk, which is the same as 2 gallons.

Mrs. Brown had 6 quarts of milk and used 2 of them. She has 4 quarts left, which is the same as 1 gallon.

$\frac{9}{2} + \frac{3}{2} = \frac{12}{2} = 6$	$\frac{9}{2} - \frac{3}{2} = \frac{6}{2} = 3$

Marge had 9 half-dollars and received 3 half-dollars for her birthday. That is a total of 12 half-dollars, which is the same as $6.00.

A nickel is worth half as much as a dime. Al had 9 nickels and spent 3 of them; so he had 6 nickels left. He changed them into 3 dimes.

$\frac{8}{10} + \frac{7}{10} = \frac{15}{10} = 1\frac{1}{2}$	$\frac{8}{10} - \frac{7}{10} = \frac{1}{10}$

Harry rode 8-tenths of a mile and walked 7-tenths of a mile further. That's a total of 15-tenths of a mile, which is a mile and a half.

It is 8-tenths of a mile to the library. After Sue had walked 7-tenths of a mile, she had 1-tenth left to walk.

Thus far, we have been adding and subtracting pairs of fractions in which the denominators are the same.)

Numerator → $\frac{7}{4} + \frac{9}{4}$ (←Multiplier)

Denominator → (←Divider)

(Of course, Alec would say that the "dividers" are the same.)

Let's consider examples of addition and subtraction of fractions in which the denominators are not alike, such as:

$\frac{1}{3} + \frac{5}{4} =$ $\frac{7}{5} - \frac{2}{3} =$ $\frac{3}{8} + \frac{5}{6} =$ ○ $\frac{4}{7} - \frac{1}{4} =$

Many fractions are equivalent to $\frac{1}{3}$. Alec would say that many pairs of machine-settings would produce the same results as $\frac{1}{3}$.

The same can be said of all fractions such as $\frac{5}{4}$, $\frac{7}{3}$, $\frac{2}{3}$, etc.

$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} = \frac{5}{15} = \frac{6}{18} = \frac{7}{21}$ etc.

$\frac{5}{4} = \frac{10}{8} = \frac{15}{12} = \frac{20}{16} = \frac{25}{20} = \frac{30}{24} = \frac{35}{28}$ etc.

To find the sum of $\frac{1}{3}$ and $\frac{5}{4}$, we scan both lists of fractions until we find one in the list for $\frac{1}{3}$ that has the same denominator as one in the list for $\frac{5}{4}$.

$$\frac{1}{3} + \frac{5}{4}$$

$$\frac{4}{12} + \frac{15}{12} = \frac{19}{12}$$

The problem has been changed to a familiar form:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

Tasks: *Pupils supply the missing entries as they read the problems and discussion.*

Purpose: To show pupils that problems in adding fractions with different denominators can be translated into problems in adding fractions with the same denominator.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: This page moves back from the abstract to the real world of milk and dollars and miles—the final reassurance that the ideas we have been developing have sensible, useful interpretations.

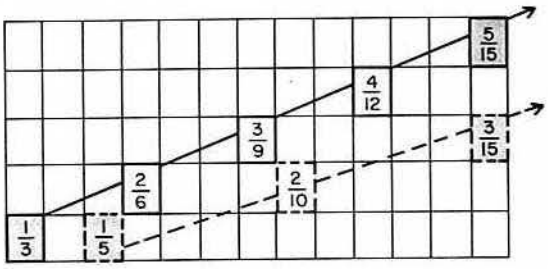
We pick up another familiar idea (see pupil pages 60 through 63, in particular)—that many pairs of operations lead to the same outcome. There are families of fractions such that each fraction in a family does precisely the same job *(is equivalent to)* as every other fraction in that family. Replacing one fraction by another from the same family does not affect the outcome, except that use of particular members helps us to change every pair of fractions into a pair of the form:

$$\frac{a}{c} \text{ and } \frac{b}{c}$$

. . . and we know a lot about adding and subtracting pairs of fractions in that particular form.

Practice sheet 76 helps dramatize the fact that there are families of fractions such that each fraction in a family can replace each of the others in the family. Each family is on a diagonal. Using different colors for the families will highlight some of the important patterns in the table.

To find the sum or difference of representatives of two different families, simply follow out the two diagonals until you find a column which contains a member of each of the families. These members have the same or *a common denominator*, and now we know how to proceed. For example:



$$\frac{1}{3} + \frac{1}{5} = \frac{5}{15} + \frac{3}{15} = \frac{8}{15}$$

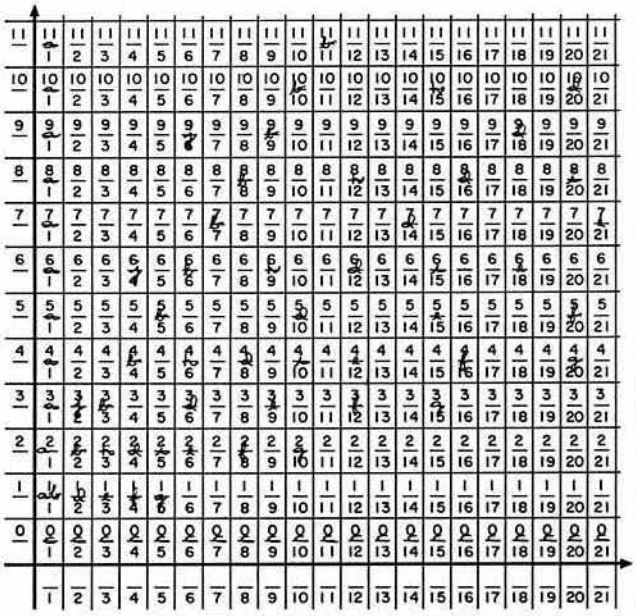
$$\frac{1}{3} - \frac{1}{5} = \frac{5}{15} - \frac{3}{15} = \frac{2}{15}$$

Some students may wish to extend this table into a kind of adding and subtracting machine for fractions.

Notice that the diagonal of the family of fractions for 1/1 (1/1, 2/2, 3/3, etc.) separates the fractions into two groups: those below that diagonal are all less than 1; those above are all greater than 1.

EPS-76

The beginning of a table:



What do you notice in the table?

a. (a) Color all cells containing the fractions that are equal to whole numbers.

b. (b) Use another color for cells containing fractions that are equal to the whole number one.

c. (c) Use another color for cells containing fractions that are equal to zero.

d. (d) Use another color for cells containing fractions equal to one-half.

(e) Select other colors for cells containing fractions equal to

$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{3}, \frac{2}{5}, \frac{3}{4}.$$

(f) If the chart were extended far enough to the right and upwards, could you think of a fraction it would *not* contain? No

(g) Egyptians wrote fractions as sums of unit fractions (fractions whose numerator is 1). You might like to experiment with this procedure:

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4} \text{ or } \frac{3}{4} = \frac{1}{3} + \frac{1}{4} + \frac{1}{6} \text{ etc.}$$

$$\frac{7}{5} - \frac{2}{3} = ?$$

a. $\frac{7}{5}$ | $\frac{14}{10}$ | $\frac{21}{15}$ | $\frac{28}{20}$ | $\frac{35}{25}$ | $\frac{42}{30}$ | $\frac{49}{35}$ | $\frac{56}{40}$ | etc.

b. $\frac{2}{3}$ | $\frac{4}{6}$ | $\frac{6}{9}$ | $\frac{8}{12}$ | $\frac{10}{15}$ | $\frac{12}{18}$ | $\frac{14}{21}$ | $\frac{16}{24}$ | etc.

c. $\frac{7}{5} - \frac{2}{3} = \frac{21}{15} - \frac{10}{15} = \frac{11}{15}$

The lists below can be used to help solve the problems that follow.

d. $\frac{3}{8}$ | $\frac{6}{16}$ | $\frac{9}{24}$ | $\frac{12}{32}$ | $\frac{15}{40}$ | $\frac{18}{48}$ | $\frac{21}{56}$ | $\frac{24}{64}$ | —

e. $\frac{5}{6}$ | $\frac{10}{12}$ | $\frac{15}{18}$ | $\frac{20}{24}$ | $\frac{25}{30}$ | $\frac{30}{36}$ | $\frac{35}{42}$ | $\frac{40}{48}$ | —

f. $\frac{4}{7}$ | $\frac{8}{14}$ | $\frac{12}{21}$ | $\frac{16}{28}$ | $\frac{20}{35}$ | $\frac{24}{42}$ | $\frac{28}{49}$ | $\frac{32}{56}$ | —

g. $\frac{1}{4}$ | $\frac{2}{8}$ | $\frac{3}{12}$ | $\frac{4}{16}$ | $\frac{5}{20}$ | $\frac{6}{24}$ | $\frac{7}{28}$ | $\frac{8}{32}$ | —

h. $\frac{7}{12}$ | $\frac{14}{24}$ | $\frac{21}{36}$ | $\frac{28}{48}$ | $\frac{35}{60}$ | $\frac{42}{72}$ | $\frac{49}{84}$ | $\frac{56}{96}$ | —

i. $\frac{8}{9}$ | $\frac{16}{18}$ | $\frac{24}{27}$ | $\frac{32}{36}$ | $\frac{40}{45}$ | $\frac{48}{54}$ | $\frac{56}{63}$ | $\frac{64}{72}$ | —

j. $\frac{3}{8} + \frac{5}{6} = \frac{9}{24} + \frac{20}{24} = \frac{29}{24}$

k. $\frac{4}{7} - \frac{1}{4} = \frac{16}{28} - \frac{7}{28} = \frac{9}{28}$

l. $\frac{8}{9} + \frac{7}{12} = \frac{32}{36} + \frac{21}{36} = \frac{53}{36}$

m. $\frac{5}{6} - \frac{4}{7} = \frac{35}{42} - \frac{24}{42} = \frac{11}{42}$

Whole numbers can be written as fractions.

$3 = 3 \div 1 = \frac{3}{1}$ also $3 = 6 \div 2 = \frac{6}{2}$ etc.

n. $3 = \frac{3}{1} = \frac{6}{2} = \frac{9}{3} = \frac{12}{4} = \frac{15}{5}$ = etc.

o. 1 | $\frac{1}{1}$ | $\frac{2}{2}$ | $\frac{3}{3}$ | $\frac{4}{4}$ | $\frac{5}{5}$ | $\frac{6}{6}$ | $\frac{7}{7}$

p. 8 | $\frac{8}{1}$ | $\frac{16}{2}$ | $\frac{24}{3}$ | $\frac{32}{4}$ | $\frac{40}{5}$ | $\frac{48}{6}$ | $\frac{56}{7}$

We use a shorthand that combines a whole number and a fraction:

$3\frac{1}{2}$ means $3 + \frac{1}{2}$ $\quad$ $8\frac{1}{3}$ means $8 + \frac{1}{3}$

q. $1\frac{4}{7}$ means $1 + \frac{4}{7}$ $\quad$ r. $3\frac{5}{6}$ means $3 + \frac{5}{6}$

We can rewrite such shorthand expressions as single fractions in this way:

s. $3\frac{3}{8} = 3 + \frac{3}{8} = \frac{3}{1} + \frac{3}{8} = \frac{24}{8} + \frac{3}{8} = \frac{27}{8}$

t. $8\frac{1}{2} = 8 + \frac{1}{2} = \frac{8}{1} + \frac{1}{2} = \frac{16}{2} + \frac{1}{2} = \frac{17}{2}$

u. $1\frac{2}{3} = 1 + \frac{2}{3} = \frac{1}{1} + \frac{2}{3} = \frac{3}{3} + \frac{2}{3} = \frac{5}{3}$

v. $8\frac{1}{2}$ | $\frac{17}{2}$ | $\frac{34}{4}$ | $\frac{51}{6}$ | $\frac{68}{8}$ | $\frac{85}{10}$ | $\frac{102}{12}$ | —

w. $1\frac{2}{3}$ | $\frac{5}{3}$ | $\frac{10}{6}$ | $\frac{15}{9}$ | $\frac{20}{12}$ | $\frac{25}{15}$ | $\frac{30}{18}$ | —

x. $8\frac{1}{2} + 1\frac{2}{3} = \frac{51}{6} + \frac{10}{6} = \frac{61}{6}$

y. $8\frac{1}{2} - 1\frac{2}{3} = \frac{51}{6} - \frac{10}{6} = \frac{41}{6}$

There is an alternate approach that often saves much work. (See next page.)

77 Exploring mixed-number notation

Tasks: *Pupils supply the missing entries.*

Purpose: To give practice in adding and subtracting fractions with different denominators, to demonstrate fraction notation for whole numbers and for "mixed numbers" (indicated sums of whole numbers and fractions), and to give practice in one method of adding and subtracting mixed numbers.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Can every pair of fractions such as:

$$\frac{7}{5} \text{ and } \frac{2}{3} \qquad \frac{3}{8} \text{ and } \frac{5}{6} \qquad \frac{8}{9} \text{ and } \frac{9}{12}$$

be written in the form of:

$$\frac{a}{c} \text{ and } \frac{b}{c}$$

—that is, be transformed into a pair of fractions with the same denominator? Let the class discuss the question.

How can you be sure? Consider $\frac{8}{9}$ and $\frac{6}{12}$. We can write down the members of the families in this way:

$$\begin{array}{ccccccccc}
 & (1) & & (2) & & (3) & & (4) & & (12) \\
\frac{8}{9} & = & \frac{8\times2}{9\times2} & = & \frac{8\times3}{9\times3} & = & \frac{8\times4}{9\times4} & \cdots & \frac{8\times12}{9\times12} & \leftarrow \\
\frac{6}{12} & = & \frac{6\times2}{12\times2} & = & \frac{6\times3}{12\times3} & = & \frac{6\times4}{12\times4} & \cdots & \frac{6\times9}{12\times9} & \leftarrow \\
 & (1) & & (2) & & (3) & & (4) & & (9)
\end{array}$$

and we know that $9\times12=12\times9$. This situation could be repeated no matter what two fractions are considered (although we might have to extend the lists quite far).

So, we can write:

$$\frac{8}{9}+\frac{6}{12}=\frac{8\times12}{9\times12}+\frac{6\times9}{12\times9}=\frac{(8\times12)+(6\times9)}{9\times12} \quad \text{or} \quad (12\times9)$$

In a more general way, suppose that a, b, c, and d are whole numbers such that $b\neq0$ and $d\neq0$ and that we wish to find the sum

$$\frac{a}{b}+\frac{c}{d}.$$

We proceed to write out the family lists.

$$\frac{a}{b}=\frac{a\times2}{b\times2}=\frac{a\times3}{b\times3}=\frac{a\times4}{b\times4}\cdots\cdots\frac{a\times d}{b\times d}$$

$$\frac{c}{d}=\frac{c\times2}{d\times2}=\frac{c\times3}{d\times3}=\frac{c\times4}{d\times4}\cdots\cdots\frac{c\times b}{d\times b}$$

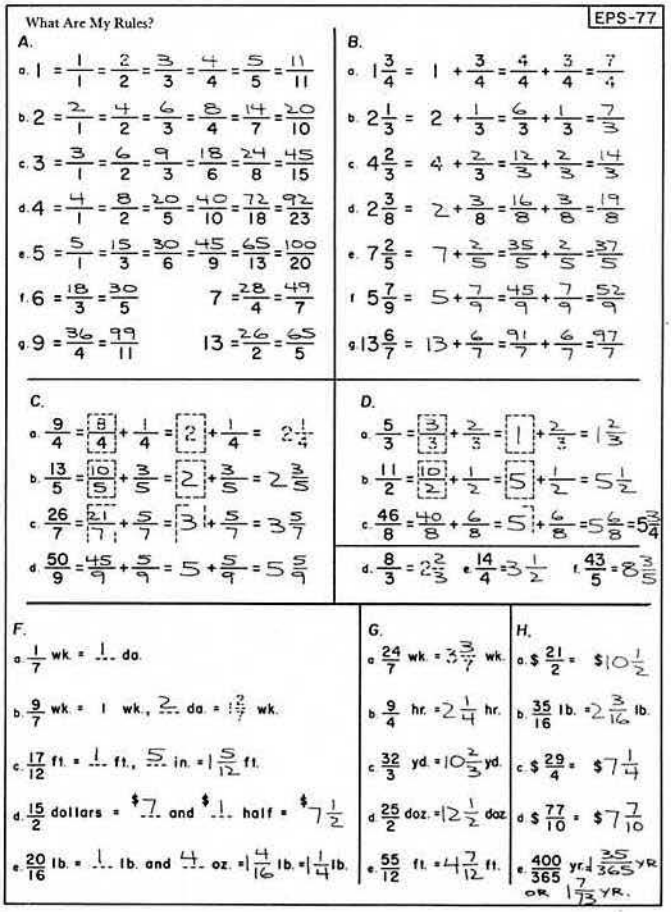
The dth entry in the list for a/b has the same denominator as the bth entry in the list for c/d because $b\times d=d\times b$. So,

$$\frac{a}{b}+\frac{c}{d}=\frac{a\times d}{b\times d}+\frac{c\times b}{a\times b}=\frac{(a\times d)+(c\times b)}{b\times d} \quad \text{or} \quad (d\times b).$$

So (using "ad" as shorthand for "$a\times d$"), we can state that, for all whole numbers a, b, c, and d such that $b\neq0$ and $d\neq0$,

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$

While this is not always the most efficient approach, it can always be carried out. Have pupils test it with several sets of four whole numbers picked at random (except that neither b nor d can be 0).



Please supply the missing entries.

A.			B.			C.			D.		
$8\frac{1}{5}$	$\rightarrow 8+\frac{1}{5}$		$3\frac{2}{9}$	$\rightarrow 3+\frac{2}{9}$		$4\frac{2}{3}$	$\rightarrow 4+\frac{2}{3}$		$13\frac{6}{7}$	$\rightarrow 13+\frac{6}{7}$	
$+2\frac{3}{5}$	$\rightarrow 2+\frac{3}{5}$		$+1\frac{5}{9}$	$\rightarrow 1+\frac{5}{9}$		$-1\frac{1}{3}$	$\rightarrow 1+\frac{1}{3}$		$-9\frac{2}{7}$	$\rightarrow 9+\frac{2}{7}$	
$10\frac{4}{5}$	$\leftarrow 10+\frac{4}{5}$		$4\frac{7}{9}$	$\leftarrow 4+\frac{7}{9}$		$3\frac{1}{3}$	$\leftarrow 3+\frac{1}{3}$		$4\frac{4}{7}$	$\leftarrow 4+\frac{4}{7}$	

Sometimes, an additional step will reduce the results to a simpler form.

E.			F.			G.		
$5\frac{1}{9}$	$\rightarrow 5+\frac{1}{9}$		$2\frac{4}{5}$	$\rightarrow 2+\frac{4}{5}$		$1\frac{3}{4}$	$\rightarrow 1+\frac{3}{4}$	
$+3\frac{2}{9}$	$\rightarrow 3+\frac{2}{9}$		$+1\frac{3}{5}$	$\rightarrow 1+\frac{3}{5}$		$+6\frac{3}{4}$	$\rightarrow 6+\frac{3}{4}$	
$8\frac{1}{3}$	$\leftarrow 8+\frac{1}{3}$	$\leftarrow 8+\frac{3}{9}$	$4\frac{2}{5}$	$\leftarrow 3+1\frac{2}{5}$	$\leftarrow 3+\frac{7}{5}$	$8\frac{1}{2}$	$\leftarrow 7+1\frac{1}{2}$	$\leftarrow 7+\frac{6}{4}$

H.			I.			J.		
$5\frac{3}{4}$	$\rightarrow 5+\frac{3}{4}$		$3\frac{7}{8}$	$\rightarrow 3+\frac{7}{8}$		$2\frac{5}{9}$	$\rightarrow 2+\frac{5}{9}$	
$-2\frac{1}{4}$	$\rightarrow 2+\frac{1}{4}$		$-1\frac{1}{8}$	$\rightarrow 1+\frac{1}{8}$		$+3\frac{7}{9}$	$\rightarrow 3+\frac{7}{9}$	
$3\frac{1}{2}$	$\leftarrow 3+\frac{1}{2}$	$\leftarrow 3+\frac{2}{4}$	$2\frac{3}{4}$	$\leftarrow 2+\frac{3}{4}$	$\leftarrow 2+\frac{6}{8}$	$6\frac{1}{3}$	$\leftarrow 5+1\frac{1}{3}$	$\leftarrow 5+\frac{12}{9}$

Sometimes, a new problem arises in a subtraction example.

K.				L.			
$5\frac{1}{4}$	$\rightarrow 4+1+\frac{1}{4}$	$\rightarrow 4+\frac{5}{4}$		$7\frac{3}{8}$	$\rightarrow 6+1+\frac{3}{8}$	$\rightarrow 6+\frac{11}{8}$	
$-1\frac{3}{4}$	$\rightarrow$	$1+\frac{3}{4}$		$-6\frac{5}{8}$	$\rightarrow$	$6+\frac{5}{8}$	
$3\frac{1}{2}$	$\leftarrow 3+\frac{1}{2}$	$\leftarrow 3+\frac{2}{4}$		$\frac{3}{4}$	$\leftarrow 0+\frac{3}{4}$	$\leftarrow 0+\frac{6}{8}$	

Use scratch paper if necessary to find the results.

M.	N.	O.	P.	Q.	R.	S.
$3\frac{1}{7}$	$9\frac{1}{4}$	$8\frac{3}{5}$	$5\frac{1}{6}$	$197\frac{2}{3}$	$3\frac{8}{9}$	$15\frac{5}{8}$
$+2\frac{4}{7}$	$+5\frac{1}{4}$	$+9\frac{4}{5}$	$+3\frac{5}{6}$	$+18\frac{2}{3}$	$+1\frac{7}{9}$	$+28\frac{7}{8}$
$5\frac{5}{7}$	$14\frac{1}{2}$	$18\frac{2}{5}$	9	$216\frac{1}{3}$	$5\frac{2}{3}$	$44\frac{1}{2}$

T.	U.	V.	W.	X.	Y.	Z.
$10\frac{5}{9}$	$38\frac{3}{4}$	$4\frac{5}{6}$	$8\frac{1}{3}$	$21\frac{1}{8}$	$14\frac{2}{5}$	$18\frac{1}{6}$
$-3\frac{1}{9}$	$-8\frac{1}{4}$	$-1\frac{1}{6}$	$-5\frac{2}{3}$	$-7\frac{3}{8}$	$-3\frac{4}{5}$	$-9\frac{5}{6}$
$7\frac{4}{9}$	$30\frac{1}{2}$	$3\frac{2}{3}$	$2\frac{2}{3}$	$13\frac{3}{4}$	$10\frac{3}{5}$	$8\frac{1}{3}$

78-79 Exploring the usual algorithms for adding and subtracting mixed numbers

Tasks: *Pupils supply the missing entries.*

Purpose: To provide practice in using common algorithms for adding and subtracting mixed numbers.

Unifying Ideas: Structure; numeration; addition and subtraction.

The Lesson: Addition and subtraction of mixed numbers can always be carried out in the ways we have been discussing, by rewriting each mixed number as a fraction. However, this is very often an inefficient way to go about it.

On pupil pages 78 and 79, we have deliberately used a rather cumbersome means of bookkeeping. Our purpose is to reveal as completely feasible one line of reasoning. We do not expect pupils to keep such complete records.

On the other hand, we do not wish to specify any particular form for them to use. There are reasons for this: (a) the pupils might mistake the form as the important development rather than the idea, and (b) we do not wish to suggest a form that is unfamiliar to the teacher. We hope that most pupils will quickly invent or adopt their own shorthand to help them keep track. Some pupils may need to be told to use a particular form, and in that case the form which you find familiar is as adequate for the pupil as it is for you.

The point is that the form pupils use is of little importance. It is only a method of keeping records of a line of reasoning. What is important is that the line of reasoning be sound. And the basic rule is: when in doubt, write down every step you take so that you can make a judgment about what you are saying.

Practice sheet 78 provides practice in finding common denominators with a bit of puzzle added by the scantiness of the information that is provided.

The *What can you say?* problem may be approached like this:

Clair's number is ⅓ of Jane's—so Jane's number must be a multiple of 3. Don's number is ¾ of Clair's number, so Clair's number must be a multiple of 4. So, Jane's number must be three times a multiple of 4—and 12 is the only number less than 20 that qualifies. The rest of the problem is very easy.

EPS-78

Rewriting expressions in the form of $\frac{a}{c}+\frac{b}{c}$ or $\frac{a}{c}-\frac{b}{c}$:

a. $\frac{2}{3}+\frac{1}{4}=\frac{8}{12}+\frac{3}{12}=\frac{16}{24}+\frac{6}{24}=\frac{40}{60}+\frac{15}{60}=\frac{80}{120}+\frac{30}{120}=\frac{48}{72}+\frac{18}{72}=\frac{11}{12}$

b. $\frac{3}{2}+\frac{5}{4}=\frac{6}{4}+\frac{5}{4}=\frac{24}{16}+\frac{20}{16}=\frac{60}{40}+\frac{50}{40}=\frac{150}{100}+\frac{125}{100}=\frac{84}{56}+\frac{70}{56}=\frac{11}{4}=2\frac{3}{4}$

c. $\frac{1}{2}-\frac{1}{6}=\frac{3}{6}-\frac{1}{6}=\frac{9}{18}-\frac{3}{18}=\frac{27}{54}-\frac{9}{54}=\frac{39}{78}-\frac{13}{78}=\frac{45}{90}-\frac{15}{90}=\frac{3}{9}=\frac{1}{3}$

d. $\frac{7}{3}-\frac{3}{5}=\frac{35}{15}-\frac{9}{15}=\frac{105}{45}-\frac{27}{45}=\frac{210}{90}-\frac{54}{90}=\frac{140}{60}-\frac{36}{60}=\frac{175}{75}-\frac{45}{75}=\frac{26}{15}=1\frac{11}{15}$

e. $\frac{5}{6}+\frac{2}{9}=\frac{15}{18}+\frac{4}{18}=\frac{150}{180}+\frac{40}{180}=\frac{45}{54}+\frac{12}{54}=\frac{60}{72}+\frac{16}{72}=\frac{75}{90}+\frac{20}{90}=\frac{19}{18}=1\frac{1}{18}$

f. $\frac{7}{8}-\frac{2}{3}=\frac{21}{24}-\frac{16}{24}=\frac{42}{48}-\frac{32}{48}=\frac{105}{120}-\frac{80}{120}=\frac{84}{96}-\frac{64}{96}=\frac{63}{72}-\frac{48}{72}=\frac{5}{24}$

g. $\frac{4}{9}+\frac{2}{15}=\frac{20}{45}+\frac{6}{45}=\frac{200}{450}+\frac{60}{450}=\frac{100}{225}+\frac{30}{225}=\frac{40}{90}+\frac{12}{90}=\frac{80}{180}+\frac{24}{180}=\frac{26}{45}$

Rewrite mixed numbers as common fractions — and then follow the pattern used above.

h. $1\frac{1}{2}+1\frac{3}{5}=\frac{3}{2}+\frac{8}{5}=\frac{15}{10}+\frac{16}{10}=\frac{75}{50}+\frac{80}{50}=\frac{105}{70}+\frac{112}{70}=\frac{60}{40}+\frac{64}{40}=\frac{31}{10}=3\frac{1}{10}$

i. $2\frac{1}{3}+\frac{4}{5}=\frac{7}{3}+\frac{4}{5}=\frac{35}{15}+\frac{12}{15}=\frac{175}{75}+\frac{60}{75}=\frac{70}{30}+\frac{24}{30}=\frac{210}{90}+\frac{72}{90}=\frac{47}{15}=3\frac{2}{15}$

j. $1\frac{3}{4}-\frac{5}{8}=\frac{7}{4}-\frac{5}{8}=\frac{14}{8}-\frac{5}{8}=\frac{42}{24}-\frac{15}{24}=\frac{70}{40}-\frac{25}{40}=\frac{112}{64}-\frac{40}{64}=\frac{9}{8}=1\frac{1}{8}$

k. $3\frac{2}{3}-1\frac{3}{4}=\frac{11}{3}-\frac{7}{4}=\frac{44}{12}-\frac{21}{12}=\frac{132}{36}-\frac{63}{36}=\frac{220}{60}-\frac{105}{60}=\frac{352}{96}-\frac{168}{96}=\frac{23}{12}=1\frac{11}{12}$

l.

What Can You Say?

In a game of find our secret numbers, a team of five students selected numbers less than 20. Then they gave the following clues:

(1) Clair said: "Mine is ¼ as large as Jane's number."

(2) Don said: "Mine is ¾ as large as Clair's number."

(3) Susan said: "Mine is ⅔ as large as Jane's number."

(4) Pete said: "Mine is 2½ times as large as Clair's number."

Are there enough clues to be sure of any of the five numbers? Which ones?

THERE ARE ENOUGH CLUES TO KNOW ALL THE NUMBERS.

CLAIRE - 4 SUSAN - 8

JANE - 12 DON - 3

PETE - 10

Denominators of fractions may be different:

A. $1\frac{1}{2} \rightarrow 1\frac{2}{4}$ $+4\frac{1}{4} \rightarrow 4\frac{1}{4}$ $5\frac{3}{4} \leftarrow 5\frac{3}{4}$	B. $4\frac{1}{6} \rightarrow 4\frac{1}{6}$ $+2\frac{1}{3} \rightarrow 2\frac{2}{6}$ $6\frac{1}{2} \leftarrow 6\frac{3}{6}$	C. $7\frac{1}{2} \rightarrow 7\frac{3}{6}$ $+1\frac{1}{3} \rightarrow 1\frac{2}{6}$ $8\frac{5}{6} \leftarrow 8\frac{5}{6}$	D. $5\frac{3}{4} \rightarrow 5\frac{6}{8}$ $+5\frac{3}{8} \rightarrow 5\frac{3}{8}$ $11\frac{1}{8} \leftarrow 10\frac{9}{8}$
E. $7\frac{3}{8} \rightarrow 7\frac{3}{8}$ $-3\frac{1}{4} \rightarrow 3\frac{2}{8}$ $4\frac{1}{8} \leftarrow 4\frac{1}{8}$	F. $8\frac{2}{3} \rightarrow 8\frac{4}{6}$ $-5\frac{1}{2} \rightarrow 5\frac{3}{6}$ $3\frac{1}{6} \leftarrow 3\frac{1}{6}$	G. $4\frac{5}{6} \rightarrow 4\frac{5}{6}$ $-1\frac{1}{3} \rightarrow 1\frac{2}{6}$ $3\frac{1}{2} \leftarrow 3\frac{3}{6}$	H. $6\frac{3}{4} \rightarrow 6\frac{9}{12}$ $-2\frac{1}{3} \rightarrow 2\frac{4}{12}$ $4\frac{5}{12} \leftarrow 4\frac{5}{12}$
I. $9\frac{1}{2} \rightarrow 9\frac{2}{4} \rightarrow 8\frac{6}{4}$ $-2\frac{3}{4} \rightarrow 2\frac{3}{4} \rightarrow 2\frac{3}{4}$ $6\frac{3}{4} \leftarrow 6\frac{3}{4}$	J. $8\frac{1}{3} \rightarrow 8\frac{2}{6} \rightarrow 7\frac{8}{6}$ $-1\frac{5}{6} \rightarrow 1\frac{5}{6} \rightarrow 1\frac{5}{6}$ $6\frac{1}{2} \leftarrow 6\frac{3}{6}$	K. $6\frac{1}{3} \rightarrow 6\frac{4}{12} \rightarrow 5\frac{16}{12}$ $-2\frac{3}{4} \rightarrow 2\frac{9}{12} \rightarrow 2\frac{9}{12}$ $3\frac{7}{12} \leftarrow 3\frac{7}{12}$	
L. $10\frac{3}{8} \rightarrow 10\frac{9}{24} \rightarrow 9\frac{33}{24}$ $-9\frac{2}{3} \rightarrow 9\frac{16}{24} \rightarrow 9\frac{16}{24}$ $\frac{17}{24} \leftarrow 0\frac{17}{24}$	M. $5\frac{1}{5} \rightarrow 5\frac{7}{35} \rightarrow 4\frac{42}{35}$ $-\frac{4}{7} \rightarrow 0\frac{20}{35} \rightarrow 0\frac{20}{35}$ $4\frac{22}{35} \leftarrow 4\frac{22}{35}$	N. $15\frac{1}{6} \rightarrow 15\frac{3}{18} \rightarrow 14\frac{21}{18}$ $-7\frac{7}{9} \rightarrow 7\frac{14}{18} \rightarrow 7\frac{14}{18}$ $7\frac{7}{18} \leftarrow 7\frac{7}{18}$	

You may wish to use the blank spaces below for scratch work to get your results.

O. $7\frac{2}{3} \rightarrow 7\frac{8}{12}$ $+2\frac{3}{4} \rightarrow 2\frac{9}{12}$ $10\frac{5}{12} \leftarrow 9\frac{17}{12}$	P. $15\frac{3}{10} \rightarrow 15\frac{12}{40} \rightarrow 14\frac{52}{40}$ $-7\frac{3}{4} \rightarrow 7\frac{30}{40} \rightarrow 7\frac{30}{40}$ $7\frac{11}{20} \leftarrow 7\frac{22}{40}$	Q. $12\frac{1}{6} \rightarrow 12\frac{3}{8} \rightarrow 11\frac{21}{18}$ $-5\frac{4}{9} \rightarrow 5\frac{8}{18} \rightarrow 5\frac{8}{18}$ $6\frac{13}{18} \leftarrow 6\frac{13}{18}$
R. $9\frac{1}{3} \rightarrow 9\frac{5}{15}$ $+9\frac{2}{5} \rightarrow 9\frac{6}{15}$ $18\frac{11}{15} \leftarrow 18\frac{11}{15}$	S. $13\frac{7}{8} \rightarrow 13\frac{21}{24}$ $+7\frac{5}{6} \rightarrow 7\frac{20}{24}$ $21\frac{17}{24} \leftarrow 20\frac{41}{24}$	T. $8\frac{3}{4} \rightarrow 8\frac{21}{28} \rightarrow 7\frac{49}{28}$ $-\frac{6}{7} \rightarrow \frac{24}{28} \rightarrow \frac{24}{28}$ $7\frac{25}{28} \leftarrow 7\frac{25}{28}$

Notes on Pupil Page 79

Tasks: *Pupils carry out the addition and subtraction problems.*

Purpose: To provide practice in using common algorithms for adding and subtracting mixed numbers involving fractions with different denominators.

Unifying Ideas: Structure; numeration; addition and subtraction.

The Lesson: Most of the subtraction problems on this page involve the idea of exchanging or "borrowing," since, in the most straightforward approach, changing to the form:

$$\frac{a}{c} - \frac{b}{c}$$

presents us with an a that is smaller than b. To overcome this difficulty, we follow exactly the same reasoning we use with whole numbers:

$$\begin{array}{r} 75 \\ -38 \\ \hline \end{array} \rightarrow \begin{array}{l} 70+5 \\ 30+8 \\ \hline \end{array} \rightarrow \begin{array}{l} 60+15 \\ 30+8 \\ \hline 30+7 \end{array}$$

$$\begin{array}{r} 9\frac{1}{2} \\ -2\frac{3}{4} \\ \hline \end{array} \rightarrow \begin{array}{l} 9+\frac{2}{4} \\ 2+\frac{3}{4} \\ \hline \end{array} \rightarrow \begin{array}{l} 8+\frac{6}{4} \\ 2+\frac{3}{4} \\ \hline 6+\frac{3}{4} \end{array}$$

It is most helpful for pupils to realize that we are really using a most familiar method to get around a familiar obstacle—that nothing fundamentally new has been introduced.

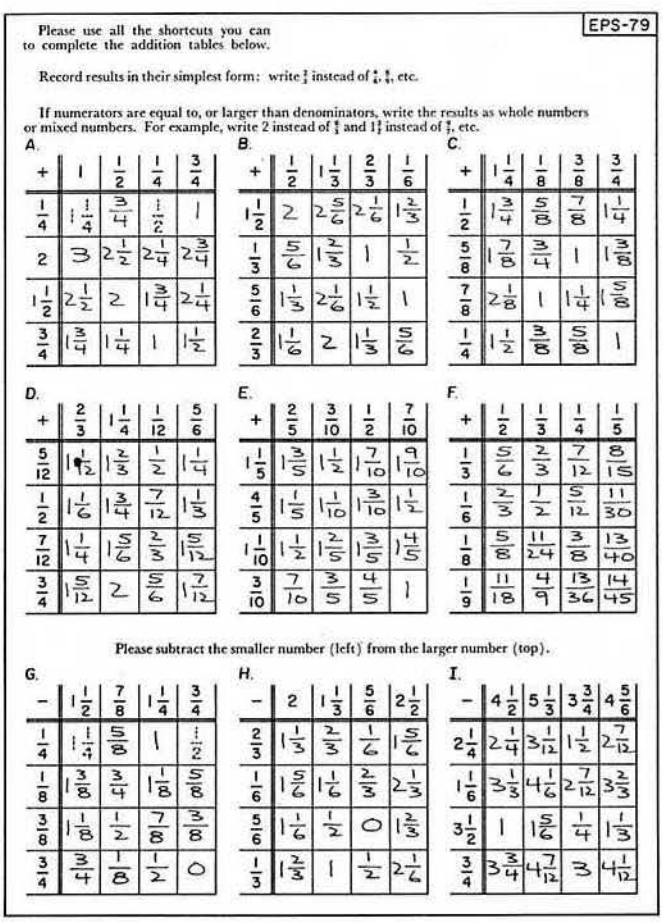
If several examples of subtraction of whole numbers involving the idea of exchanging or "borrowing" are written out in full as in the example above, two purposes will be accomplished: (a) the similarity of this approach with the one used on pupil page 79 will become clear, and (b) the pupils will be encouraged to find a simpler method of bookkeeping—as they did in subtraction of whole numbers.

You may strengthen these ideas by considering addition in the same way when the idea of exchanging or "carrying" is used:

$$\begin{array}{r} 26 \\ +49 \\ \hline \end{array} \rightarrow \begin{array}{l} 20+6 \\ 40+9 \\ \hline 60+15 \end{array} = (60+10)+5$$

$$\begin{array}{r} 2\frac{3}{4} \\ +1\frac{7}{8} \\ \hline \end{array} \quad \begin{array}{l} 2+\frac{6}{8} \\ 1+\frac{7}{8} \\ \hline 3+\frac{13}{8} \end{array} = (3+1)+\frac{5}{8}$$

Practice sheet 79 may be used as a checkup of each pupil's fluency with more or less simple examples of addition and subtraction of fractions. Encourage efforts to write down results without using scratch paper. On the other hand, when there is any doubt, encourage the use of scratch paper.



EPS-79

Please use all the shortcuts you can to complete the addition tables below.

Record results in their simplest form: write $\frac{1}{2}$ instead of $\frac{2}{4}$, $\frac{1}{3}$ instead of $\frac{2}{6}$, etc.

If numerators are equal to, or larger than denominators, write the results as whole numbers or mixed numbers. For example, write 2 instead of $\frac{4}{4}$ and $1\frac{1}{2}$ instead of $\frac{3}{2}$, etc.

A.

+	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	1
2	3	$2\frac{1}{2}$	$2\frac{1}{4}$	$2\frac{3}{4}$
$1\frac{1}{2}$	$2\frac{1}{2}$	2	$1\frac{3}{4}$	$2\frac{1}{4}$
$\frac{3}{4}$	$1\frac{3}{4}$	$1\frac{1}{4}$	1	$1\frac{1}{2}$

B.

+	$\frac{1}{2}$	$1\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$
$1\frac{1}{2}$	2	$2\frac{5}{6}$	$2\frac{1}{6}$	$1\frac{2}{3}$
$\frac{1}{3}$	$\frac{5}{6}$	$1\frac{2}{3}$	1	$\frac{1}{2}$
$\frac{5}{6}$	$1\frac{1}{3}$	$2\frac{1}{6}$	$1\frac{1}{2}$	1
$\frac{2}{3}$	$1\frac{1}{6}$	2	$1\frac{1}{3}$	$\frac{5}{6}$

C.

+	$1\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{4}$
$\frac{1}{2}$	$1\frac{3}{4}$	$\frac{5}{8}$	$\frac{7}{8}$	$1\frac{1}{4}$
$\frac{5}{8}$	$1\frac{7}{8}$	$\frac{3}{4}$	1	$1\frac{3}{8}$
$\frac{7}{8}$	$2\frac{1}{8}$	1	$1\frac{1}{4}$	$1\frac{5}{8}$
$\frac{1}{4}$	$1\frac{1}{2}$	$\frac{3}{8}$	$\frac{5}{8}$	1

D.

+	$\frac{2}{3}$	$1\frac{1}{4}$	$\frac{1}{12}$	$\frac{5}{6}$
$\frac{5}{12}$	$1\frac{1}{2}$	$1\frac{2}{3}$	$\frac{1}{2}$	$1\frac{1}{4}$
$\frac{1}{2}$	$1\frac{1}{6}$	$1\frac{3}{4}$	$\frac{7}{12}$	$1\frac{1}{3}$
$\frac{7}{12}$	$1\frac{1}{4}$	$1\frac{5}{6}$	$\frac{2}{3}$	$1\frac{5}{12}$
$\frac{3}{4}$	$1\frac{5}{12}$	2	$\frac{5}{6}$	$1\frac{7}{12}$

E.

+	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{7}{10}$
$1\frac{1}{5}$	$1\frac{3}{5}$	$1\frac{1}{2}$	$1\frac{7}{10}$	$1\frac{9}{10}$
$\frac{4}{5}$	$1\frac{1}{5}$	$1\frac{1}{10}$	$1\frac{3}{10}$	$1\frac{1}{2}$
$1\frac{1}{10}$	$1\frac{1}{2}$	$1\frac{2}{5}$	$1\frac{3}{5}$	$1\frac{4}{5}$
$\frac{3}{10}$	$\frac{7}{10}$	$\frac{3}{5}$	$\frac{4}{5}$	1

F.

+	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
$\frac{1}{3}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{8}{15}$
$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{11}{30}$
$\frac{1}{8}$	$\frac{5}{8}$	$\frac{11}{24}$	$\frac{3}{8}$	$\frac{13}{40}$
$\frac{1}{9}$	$\frac{11}{18}$	$\frac{4}{9}$	$\frac{13}{36}$	$\frac{14}{45}$

Please subtract the smaller number (left) from the larger number (top).

G.

−	$1\frac{1}{2}$	$\frac{7}{8}$	$1\frac{1}{4}$	$\frac{3}{4}$
$\frac{1}{4}$	$1\frac{1}{4}$	$\frac{5}{8}$	1	$\frac{1}{2}$
$\frac{1}{8}$	$1\frac{3}{8}$	$\frac{3}{4}$	$1\frac{1}{8}$	$\frac{5}{8}$
$\frac{3}{8}$	$1\frac{1}{8}$	$\frac{1}{2}$	$\frac{7}{8}$	$\frac{3}{8}$
$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	0

H.

−	2	$1\frac{1}{3}$	$\frac{5}{6}$	$2\frac{1}{2}$
$\frac{2}{3}$	$1\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$1\frac{5}{6}$
$\frac{1}{6}$	$1\frac{5}{6}$	$1\frac{1}{6}$	$\frac{2}{3}$	$2\frac{1}{3}$
$\frac{5}{6}$	$1\frac{1}{6}$	$\frac{1}{2}$	0	$1\frac{2}{3}$
$\frac{1}{3}$	$1\frac{2}{3}$	1	$\frac{1}{2}$	$2\frac{1}{6}$

I.

−	$4\frac{1}{2}$	$5\frac{1}{3}$	$3\frac{3}{4}$	$4\frac{5}{6}$
$2\frac{1}{4}$	$2\frac{1}{4}$	$3\frac{1}{12}$	$1\frac{1}{2}$	$2\frac{7}{12}$
$1\frac{1}{6}$	$3\frac{1}{3}$	$4\frac{1}{6}$	$2\frac{7}{12}$	$3\frac{2}{3}$
$3\frac{1}{2}$	1	$1\frac{5}{6}$	$\frac{1}{4}$	$1\frac{1}{3}$
$\frac{3}{4}$	$3\frac{3}{4}$	$4\frac{7}{12}$	3	$4\frac{1}{12}$

$\frac{1}{4} + \frac{1}{2} =$ (1) $\frac{2}{6}$ (2) $\frac{3}{4}$ (3) $\frac{1}{8}$ (4) $\frac{2}{2}$

Mark correct answer → (2) ● Cross out furthest from correct answer ← (3) ⊗

(1) ○ (2) ● (3) ⊗ (4) ○ 5 ← Difference between the answers you marked

1 2



What part of the sketch is shaded?

(1) $\frac{1}{2}$ (2) $\frac{2}{3}$ (3) $\frac{3}{5}$ (4) $\frac{3}{2}$

(1) ○ (2) ○ (3) ● (4) ⊗ $\frac{9}{10}$

$\frac{5}{6} - \frac{1}{3} =$

(1) $\frac{4}{3}$ (2) $\frac{6}{9}$ (3) $\frac{5}{18}$ (4) $\frac{1}{2}$

(1) ⊗ (2) ○ (3) ○ (4) ● $\frac{5}{6}$

3

Which fraction is more than $1\frac{1}{8}$, but less than $1\frac{1}{4}$?

(1) $1\frac{1}{6}$ (2) $1\frac{1}{2}$ (3) $1\frac{1}{3}$ (4) $1\frac{1}{10}$

(1) ● (2) ⊗ (3) ○ (4) ○ $\frac{1}{3}$

4 5

Which is the smallest fraction?

(1) $\frac{6}{9}$ (2) $\frac{8}{15}$ (3) $\frac{9}{12}$ (4) $\frac{6}{4}$

(1) ○ (2) ● (3) ○ (4) ⊗ $\frac{29}{30}$

Which fraction is equal to $3\frac{3}{5}$?

(1) $\frac{11}{5}$ (2) $\frac{6}{5}$ (3) $\frac{33}{5}$ (4) $\frac{18}{5}$

(1) ○ (2) ○ (3) ⊗ (4) ● $\frac{15}{5}$ OR 3

6

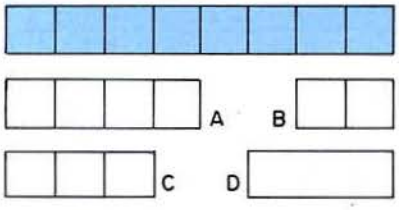
Which fraction contains 3 as a common factor of both numerator and denominator?

(1) $\frac{12}{18}$ (2) $\frac{9}{13}$ (3) $\frac{5}{15}$ (4) $\frac{10}{24}$

(1) ● (2) ○ (3) ⊗ (4) ○ $\frac{1}{3}$

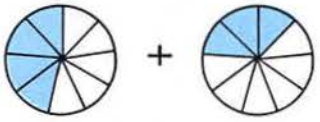
7 8

Which figure is $\frac{1}{4}$ of the tinted figure?



(1) A (2) B (3) C (4) D

(1) ⊗ (2) ● (3) ○ (4) ○ 2 BLOCKS



Which addition example is represented by the diagram?

(1) $\frac{1}{2} + \frac{1}{4}$ (3) $\frac{1}{3} + \frac{1}{4}$

(2) $\frac{4}{5} + \frac{3}{6}$ (4) $\frac{4}{9} + \frac{1}{3}$

(1) ○ (2) ⊗ (3) ○ (4) ● $\frac{47}{90}$

9

How many hours are there between 10:30 A.M. and 4:15 P.M. on the same day?

(1) $5\frac{3}{4}$ (3) $14\frac{3}{4}$

(2) $6\frac{1}{4}$ (4) $4\frac{3}{4}$

(1) ● (2) ○ (3) ⊗ (4) ○ 9 HR.

10 11

Bernie had a launching tower for a model rocket that was $18\frac{1}{4}$ inches tall. Carl, his friend, had one that was $32\frac{3}{4}$ inches tall. How many inches taller is Carl's tower?

(1) $14\frac{1}{2}$ (2) 51 (3) $18\frac{1}{2}$ (4) 14

(1) ● (2) ⊗ (3) ○ (4) ○ $36\frac{1}{2}$ IN.

Mrs. Burns had a piece of cloth $3\frac{1}{8}$ yards long. She used $1\frac{1}{4}$ yards to make a blouse. How many yards of material did she have left over?

(1) $4\frac{3}{8}$ (2) $2\frac{1}{8}$ (3) $1\frac{7}{8}$ (4) $2\frac{1}{12}$

(1) ⊗ (2) ○ (3) ● (4) ○ $2\frac{1}{2}$ YD.

12 13

A recipe designed to serve 6 calls for $3\frac{1}{2}$ cups of water. Mrs. Barnes had 9 to serve, so she increased the $3\frac{1}{2}$ cups of water to a total of how many cups?

(1) $6\frac{1}{2}$ (2) $5\frac{1}{4}$ (3) 5 (4) $4\frac{1}{4}$

(1) ⊗ (2) ● (3) ○ (4) ○ $\frac{1}{4}$ CUPS

Bill's dog eats $1\frac{1}{4}$ cans of dog food a day. The price of the dog food was 3 cans for 50¢. How much does it cost him to feed his dog for 12 days?

(1) $ 2.00 (2) $ 12.25 (3) $ 2.50 (4) $ 7.50

(1) ○ (2) ⊗ (3) ● (4) ○ $ 9.75

80 Various forms of typical multiple-choice questions with an additional twist

Tasks: *Pupils respond to the multiple-choice questions in accordance with the instructions given at the top of the page.*

Purpose: To assess pupils' achievement in computation with fractions and in solving word problems.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division.

The Lesson: Pupil page 80, prepared in the multiple-choice style of standardized achievement tests, includes examples closely comparable to the most difficult items in current tests at the seventh and eighth grade levels.

That such simple examples appear in intermediate grade tests is a clear indication that we need to use every facility available to help shed more light for pupils in working with fractions.

Standardized achievement tests ask only the first part of each question asked on this page: *What is the right answer?* In most instances, the second question is much more difficult: *What is the difference between the correct answer and the answer furthest from it?*

For this reason, you may wish to keep a raw score of the number of correct answers selected—and, perhaps, give considerable extra recognition for those examples in which all questions are answered fully.

Practice sheet 80 offers a series of examples that may help build more confidence in "manipulating" fractions.

There are patterns one can follow without any thought at all.

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \qquad \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

Often, these are precisely the steps one would take even if he were thinking his way along. However, a lot of time can be wasted if the fractions are not changed to their simplest forms. Clearly, the example:

$$\frac{25}{35}+\frac{6}{21}=\frac{25\times 21+35\times 6}{35\times 21}$$

will entail much more arduous computation than:

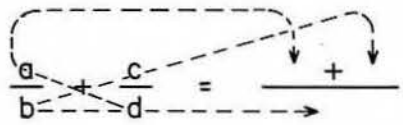
$$\frac{\cancel{25}^{5}}{\cancel{35}_{7}}+\frac{\cancel{6}^{3}}{\cancel{21}_{7}}=\frac{8}{7}=1\frac{1}{7}$$

However, it is comforting to have a tool available for all situations, no matter how foreboding it appears at first.

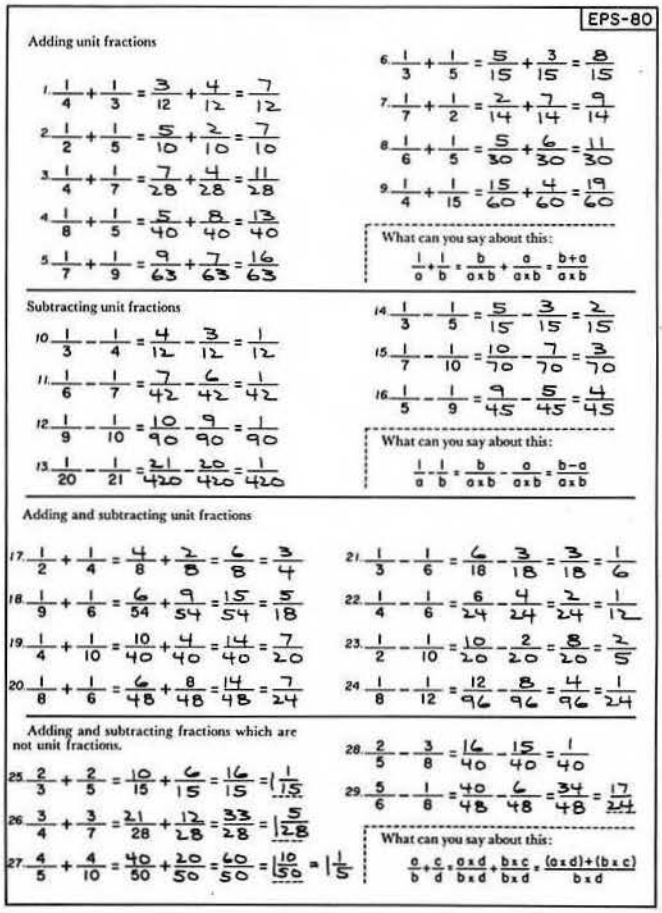
Further, there is a certain reward in noticing an obvious shortcut in adding or subtracting unit fractions: add or subtract the denominator to get the numerator of the answer, and multiply the denominators for the new denominator:

$$\frac{1}{3}+\frac{1}{7}=\frac{7+3}{3\times 7} \qquad \frac{1}{3}-\frac{1}{7}=\frac{7-3}{7\times 3}$$

Some pupils may develop a useful kinesthetic feeling about adding fractions suggested by the arrows (multiplication):

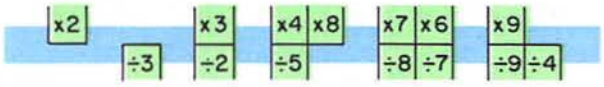


which is actually another way to state the formula.



A Page from Alec's Notebook

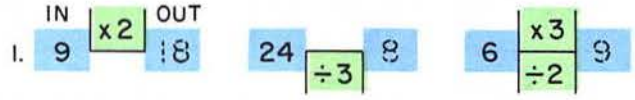
I'll refer to each of the following as a "system."



I'll experiment to find out what these systems do when I consider a number of things or a number of units of measure, such as

4 , 1 , 17 , 105 etc.

I'll record my experiments in this way:



From here on, I'll omit the '×' and '÷' signs.

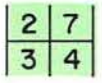
Let's agree: Numbers written in the green block above the line are multipliers; those written below the line are dividers.

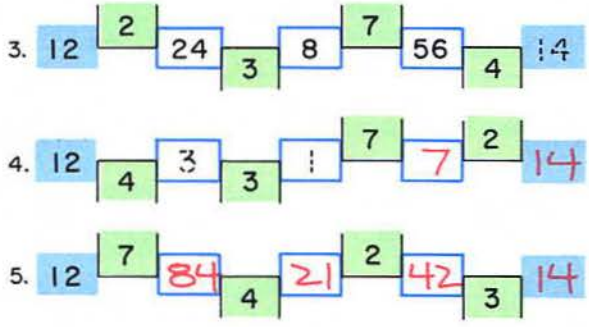
I'll not worry about the order I follow in using systems that have more than one operation.



6 x 3 = 18 ; 18 ÷ 2 = 9 6 ÷ 2 = 3 ; 3 x 3 = 9

Let's try a larger system:

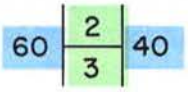




A lot of experimenting leads me to believe that the order in which I follow the directions given in a system does not affect the outcome at all.

However, in a simple 2-operation system involving both division and multiplication, I'll usually do the division first (for two reasons):

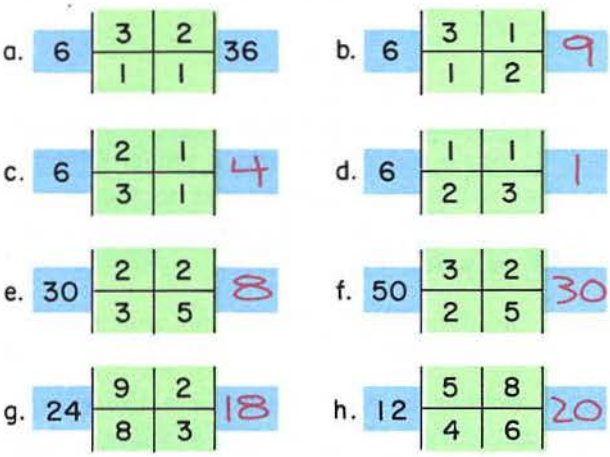
(1) I'll want my 2-operation system to behave like what people call "fractions." By $\frac{2}{3}$ of 60 minutes, people mean: one-third of 60 minutes is 20 minutes (÷ 3) and $\frac{2}{3}$ is twice as much, or 40 minutes (× 2). So, in this example:



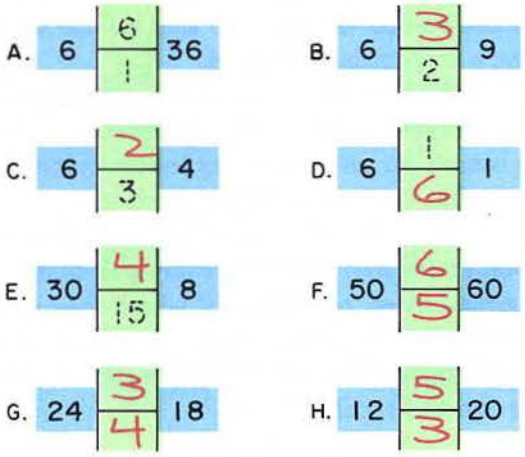
I'll divide by 3 and multiply the result by 2.

(2) It will frequently lead to simpler arithmetic.

Here are some 4-operation systems and an example of one use of each.



Notice that I can find 2-operation systems that do the same jobs as each of the above systems.



81-84 Exploring replacing 4-operation systems by 2-operation systems (Multiplication of fractions preceded by cancellation)

Tasks: *Pupils read the "story" and supply missing entries.*

Purpose: To lay the groundwork again for multiplying fractions.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: This "page from Alec's notebook" introduces the idea that fractions may be thought of as systems consisting of a pair of multiplication and division operations, one following the other. Further, he notes that a hookup of two 2-operation systems can be replaced by a single 2-operation system.

The following is a statement about fractions:

$$\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$$

Alec's more primitive way of thinking would lead him to think about applying the 4-operation system to, perhaps, 42 eggs.

42	×2	84	÷3	28	×5	140	÷7	20

So, he knows what the system would do to 42 eggs. He would shy away from the difficult computations, knowing from experiment (a) that such systems are not affected by changing the order in which the operations are applied, and (b) that carrying out the division operations first has real computational advantage. So, he would prefer this:

42	÷7	6	÷3	2	×5	10	×2	20

From further experiment, he knows that dividing by 7 and then dividing the result by 3 amounts to dividing by 21 in the first place, and that multiplying by 5 and then by 2 amounts to multiplying by 10. So, he writes:

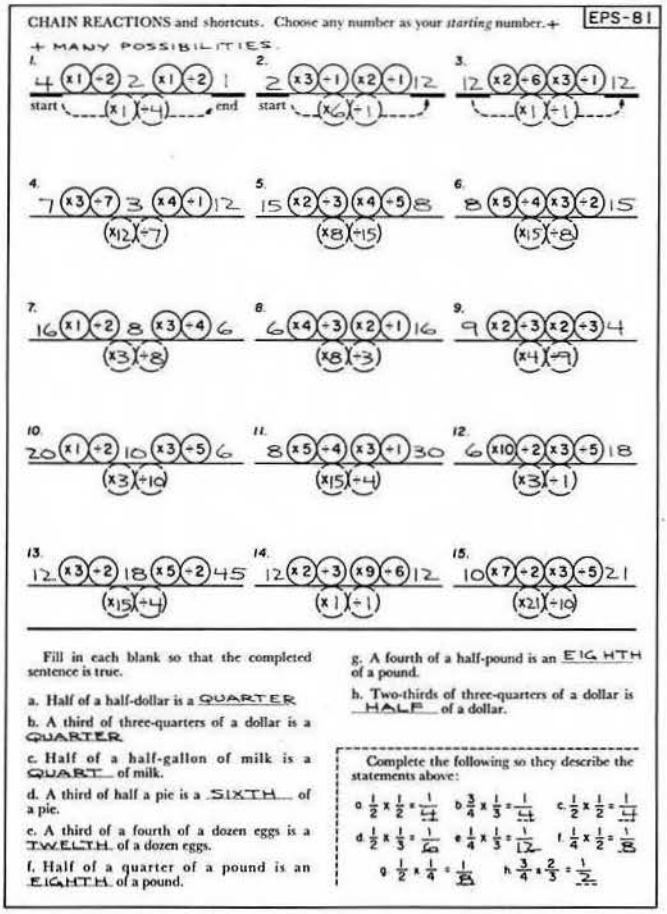
42	÷21	2	×10	20

Or, in Alec's notation:

42	$\frac{10}{21}$	20

In following Alec's increasingly less primitive attacks on the problems he takes up, pupils develop a deeper feeling of familiarity with the operation called "multiplication of fractions."

Most important in this approach is the close bond maintained with the real world. If new problems arise or if old ones need clarification, we resort to eggs and inches and dollars to see our way ahead.



CHAIN REACTIONS and shortcuts. Choose any number as your *starting* number. + EPS-81

+ MANY POSSIBILITIES.

1. 4 (×1)(÷2) 2 (×1)(÷2) 1 — start … end; (×1)(÷4)
2. 2 (×3)(÷1)(×2)(÷1) 12 — start … end; (×6)(÷1)
3. 12 (×2)(÷6)(×3)(÷1) 12; (×1)(÷1)
4. 7 (×3)(÷7) 3 (×4)(÷1) 12; (×12)(÷7)
5. 15 (×2)(÷3)(×4)(÷5) 8; (×8)(÷15)
6. 8 (×5)(÷4)(×3)(÷2) 15; (×15)(÷8)
7. 16 (×1)(÷2) 8 (×3)(÷4) 6; (×3)(÷8)
8. 6 (×4)(÷3)(×2)(÷1) 16; (×8)(÷3)
9. 9 (×2)(÷3)(×2)(÷3) 4; (×4)(÷9)
10. 20 (×1)(÷2) 10 (×3)(÷5) 6; (×3)(÷10)
11. 8 (×5)(÷4)(×3)(÷1) 30; (×15)(÷4)
12. 6 (×10)(÷2)(×3)(÷5) 18; (×3)(÷1)
13. 12 (×3)(÷2) 18 (×5)(÷2) 45; (×15)(÷4)
14. 12 (×2)(÷3)(×9)(÷6) 12; (×1)(÷1)
15. 10 (×7)(÷2)(×3)(÷5) 21; (×21)(÷10)

Fill in each blank so that the completed sentence is true.

a. Half of a half-dollar is a QUARTER
b. A third of three-quarters of a dollar is a QUARTER
c. Half of a half-gallon of milk is a QUART of milk.
d. A third of half a pie is a SIXTH of a pie.
e. A third of a fourth of a dozen eggs is a TWELTH of a dozen eggs.
f. Half of a quarter of a pound is an EIGHTH of a pound.
g. A fourth of a half-pound is an EIGHTH of a pound.
h. Two-thirds of three-quarters of a dollar is HALF of a dollar.

Complete the following so they describe the statements above:

a. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ b. $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$ c. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
d. $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ e. $\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$ f. $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$
g. $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ h. $\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

I have a hunch that I can find a 2-operation system for every 4-operation system so that both systems produce the same results.

Consider 2-operation systems in which both operations are either both multiplier machines or both divider machines.

If 2 systems always produce the same results, I call them "equal."

1. 7 [2 | 5] 70 , 7 [10] 70 , [2 | 5] = [10]

2. 5 [4 | 3] 60 , 5 [12] 60 , [4 | 3] = [12]

3. 24 [4 | 2] 3 , 24 [8] 3 , [4 | 2] = [8]

4. 45 [5 | 3] 3 , 45 [15] 3 , [5 | 3] = [15]

5. 24 [1 | 5 / 4 | 3] 10 , 24 [5 / 12] 10 , [1 | 5 / 4 | 3] = [5 / 12]

6. 45 [4 | 2 / 5 | 3] 24 , 45 [8 / 15] 24 , [4 | 2 / 5 | 3] = [8 / 15]

7. 18 [5 | 4 / 2 | 3] 60 , 18 [20 / 6] 60 , [5 | 4 / 2 | 3] = [20 / 6]

8. 60 [3 | 4 / 5 | 6] 24 , 60 [12 / 30] 24 , [3 | 4 / 5 | 6] = [12 / 30]

9. 60 [2 | 1 / 1 | 5] 24 , 60 [2 / 5] 24 , [2 | 1 / 1 | 5] = [2 / 5]

Those last two examples convince me that

[3 | 4 / 5 | 6] = [2 | 1 / 1 | 5] = [12 / 30] = [2 / 5]

Clearly, the last of the 4 equal systems is simpler than the others. How do you find simpler systems?

It may help to notice this:

10. 7 [1 / 1] 7 27 [9 / 9] 27 42 [13 / 13] 42

Here are some experiments:

11. 50 [6 / 10] 30 50 [2 | 3 / 2 | 5] 30 50 [3 / 5] 30

12. 42 [12 / 42] 12 42 [6 | 2 / 6 | 7] 12 42 [2 / 7] 12

13. 60 [16 / 20] 48 60 [4 | 4 / 4 | 5] 48 60 [4 / 5] 48

14. 72 [6 / 9] 48 72 [3 | 2 / 3 | 3] 48 72 [2 / 3] 48

If I use the largest common factor of both the divider and multiplier machines, I will find the simplest 2-operation system.

This same idea, together with the in-any-order rule, can help simplify 4-operation systems.

15. 18 [5 | 3 / 3 | 2] 45 , 18 [3 | 5 / 3 | 2] 45 , 18 [5 / 2] 45

16. 12 [4 | 7 / 1 | 4] 84 , 12 [4 | 7 / 4 | 1] 84 , 12 [7 / 1] 84

From now on, I'll use this shorthand in systems like this:

17. 24 [7 | 5 / 5 | 6] 28 , 24 [7 / 6] 28

I'll use a similar shorthand to keep track when taking out the largest common factor:

18. 24 [9 / 12] 18 (3, 4) 24 [3 / 4] 18

Sometimes I find more than one opportunity to use this idea in a single example.

19. 70 [7 | 5 / 14 | 5] 35 (1, 1, 2, 1) 70 [1 / 2] 35

(continued next page)

Tasks: *Pupils read the story and supply the missing entries.*

Purpose: To lay the groundwork for multiplying fractions, for "reducing" fractions to lowest terms, and for removing common factors before multiplying fractions.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: Perhaps the most significant notion in Alec's study of his systems is the fact that no matter how numerous the component operations become, there is always a 2-operation system that would accomplish the same end result.

This is far from obvious to the novice. And it is a rewarding thing to discover.

Yet this less-than-obvious fact is implicit in asking the question:

$$\frac{2}{5}\times\frac{3}{4}=?$$

Why don't pupils wonder whether there exists a fraction—a single numerator and a single denominator—that will serve as the answer to the above question? Or perhaps they do wonder in silence. And when we write:

$$\frac{2}{5}\times\frac{3}{4}=\frac{3}{10},$$

do pupils wonder in what respect are the things represented by the two expressions so similar that we place an equality sign between the expressions?

Alec puts an equality sign between those expressions because, when translated into operations he can carry out, no matter what quantities he decides to start with, both systems of operations produce identical end results—so much so that no one would know which system he had used.

As he becomes convinced that he can always find a 2-operation system equal to each more complicated one, the search for shortcuts has center stage. His conviction that the order in which operations are carried out does not affect the end result is most useful, permitting him to consider any pair of operations as consecutive, regardless of where they occur in the given system. Any system of multiplication and division operations can be rearranged without changing the end results.

$$\begin{vmatrix} 7 & 5 \\ 5 & 6 \end{vmatrix} = \begin{vmatrix} 5 & 7 \\ 5 & 6 \end{vmatrix} \text{ and } \begin{vmatrix} 5 \\ 5 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

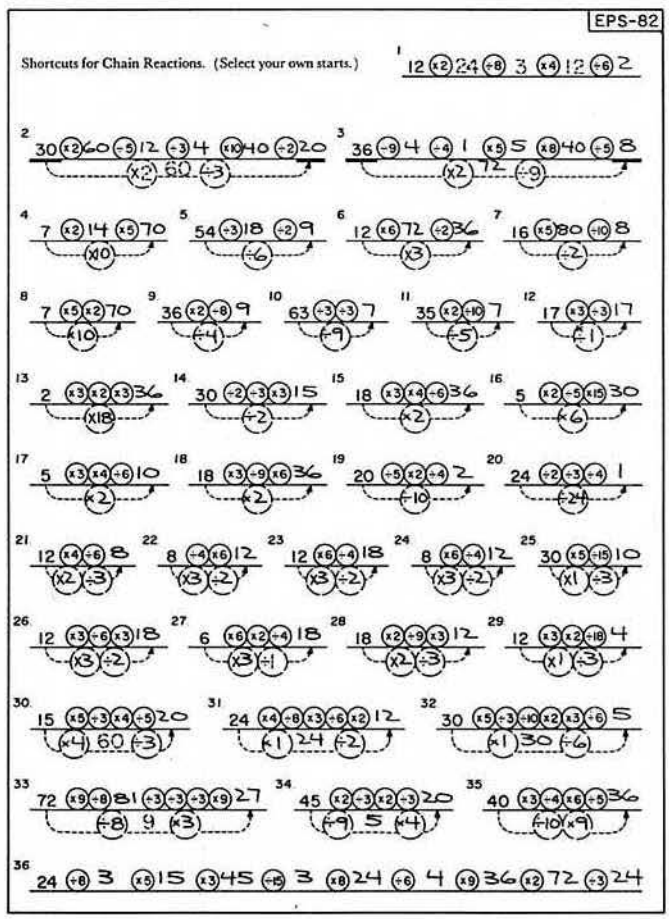
$$\begin{vmatrix} 1 & 7 \\ 1 & 6 \end{vmatrix} = \frac{7}{6} \text{ because } \begin{vmatrix} 1 & 7 \end{vmatrix} = \begin{vmatrix} 7 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 6 \end{vmatrix} = \begin{vmatrix} 6 \end{vmatrix}$$

In short:

$$\begin{vmatrix} 7 & \cancel{5}^{1} \\ \cancel{5}_{1} & 6 \end{vmatrix} = \begin{vmatrix} 7 \\ 6 \end{vmatrix}$$

"Canceling" is only a shorthand method for keeping track of the results of simplifying multiplication-division pairs within a system.

Finding shortcuts for chain reactions on practice sheet 82 leads to exactly the same kind of search for a simpler system and to the need for efficient methods of keeping records.



Here are further experiments:

1. 35 | 15 7 / 7 5 | 105 , 35 | 3 / 1 | 105

2. 30 | 24 35 / 25 18 | 56 , 30 | 28 / 15 | 56

3. 30 | 4 27 / 15 6 | 36 , 30 | 2 9 / 5 3 | 36 , 30 | 6 / 5 | 36

I will now omit the inputs and outputs (unless I am not sure that two systems are equal).

4. | 7 3 / 10 7 | = | 3 / 10 |

5. | 8 10 / 7 15 | = | 16 / 21 |

6. | 3 2 / 7 5 | = | 6 / 35 |

7. | 3 4 / 10 9 | = | 2 / 15 |

8. | 14 7 / 21 35 | = | 2 / 15 |

9. | 27 25 / 5 36 | = | 15 / 4 |

10. | 25 12 / 24 7 | = | 25 / 14 |

11. | 3 4 / 7 12 | = | 1 / 7 |

12. | 17 2 / 51 5 | = | 2 / 15 |

13. | 26 10 / 15 39 | = | 4 / 9 |

14. | 7 3 / 8 2 | = | 21 / 16 |

15. | 8 27 / 30 12 | = | 3 / 5 |

16. 32 | 7 3 / 8 2 | 42 , 32 | 21 / 16 | 42

17. 90 | 8 27 / 30 12 | 54 , 90 | 3 / 5 | 54

+ SEVERAL POSSIBILITIES.

(Note: I could make a machine that could do this kind of arithmetic.)

When people speak about "$\frac{2}{3}$ of a dozen eggs" (12 eggs), or "$\frac{1}{4}$ of a yard" (36 inches), or "$\frac{7}{8}$ of a day" (24 hours), they are thinking along the lines of my 2-operation system.

(They say)	(They write)	(My system)
18. $\frac{2}{3}$ of a dozen	$12 \times \frac{2}{3}$	12 | ×2 / ÷3 | 8
19. $\frac{1}{4}$ of a yard	$36 \times \frac{1}{4}$	36 | ×1 / ÷4 | 9
20. $\frac{7}{8}$ of a day	$24 \times \frac{7}{8}$	24 | ×7 / ÷8 | 21

People say: "The class had $\frac{2}{3}$ of an hour to do the test. Bill used up $\frac{4}{5}$ of the time allowed. How much time did Bill use?"

In finding the answer, people say: "To find $\frac{2}{3}$ of 60 minutes, I first find $\frac{1}{3}$ of 60 — $60 \div 3 = 20$. Now $\frac{2}{3}$ must be $20 \times 2 = 40$. To find $\frac{4}{5}$ of 40 minutes, I will find $\frac{1}{5}$ first: $40 \div 5 = 8$; and $\frac{4}{5}$ will be 4 times as much: $8 \times 4 = 32$. So, Bill must have used 32 minutes."

They follow the same steps I do when I use my 4-operation system with "60 minutes" as the input:

60 | 2 4 / 3 5 | 32 or 60 | 8 / 15 | 32

When people speak about "Multiplication of Fractions," such as:

$$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

they ask the same kind of questions I ask when I want to find simpler systems:

| 2 4 / 3 5 | = | 8 / 15 |

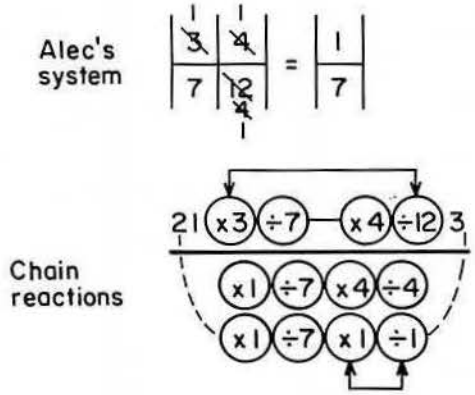
Notes on Pupil Page 83

Tasks: *Pupils read the story and supply the missing entries.*

Purpose: To lay the groundwork for multiplying fractions, for "reducing" fractions to lowest terms, and for removing common factors before multiplying fractions.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: The parallelism between Alec's systems and chain reactions is apparent as we consider pupil page 83 and practice sheet 83. Each has its own slightly different method of keeping records of simplification, but both reach the same result.



From development of such methods for sheer manipulation, we turn immediately (right-hand column on pupil page 83) to applications in which we might have some reason for finding a simpler system than

$$\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array} \quad \text{or} \quad \frac{2}{3}\times\frac{4}{5}.$$

Applications are very easy to suggest when we think as Alec thinks. Those operations will arise whenever the situation is such that you wish to find $\frac{2}{3}$ of anything, and then $\frac{4}{5}$ of that result.

I spent $\frac{2}{3}$ of my allowance; $\frac{4}{5}$ of what I spent went for a book. If my allowance is 45¢ then

$$45¢ \times \frac{2}{3} = 30¢,$$

and

$$30¢ \times \frac{4}{5} = 24¢.$$

Of the 45¢ allowance, 24¢ was spent on the book. A shortcut from 45¢ to 24¢ is: divide by 15 and multiply by 8:

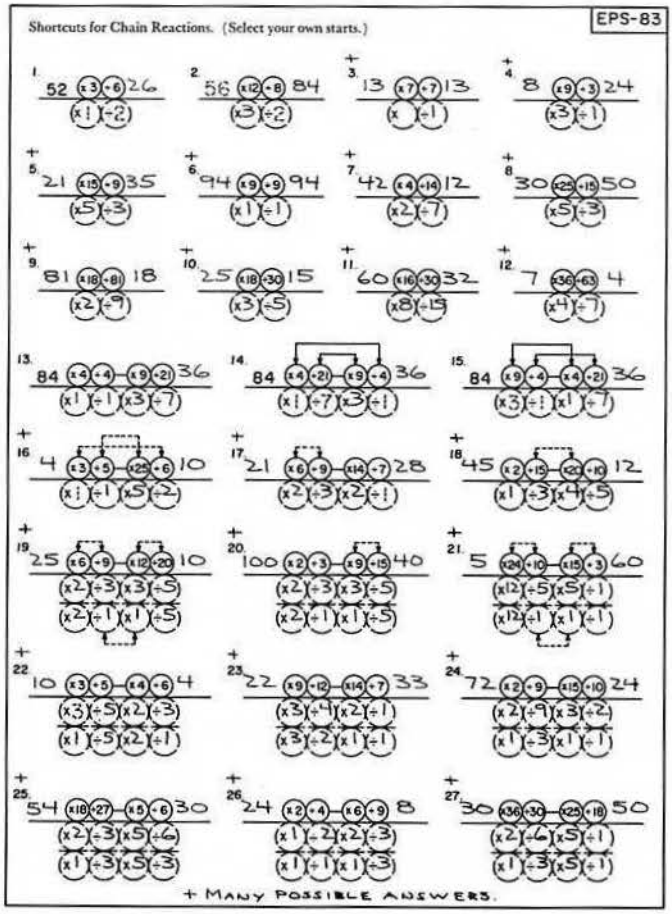
$$\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}.$$

Mrs. Brown bought $\frac{2}{3}$ of a yard of cloth. She used $\frac{4}{5}$ of what she bought to make a blouse.

Mary had an assignment to read $\frac{2}{3}$ of a 120-page book before vacation starts. She has finished $\frac{4}{5}$ of the assignment.

An application exists whenever we wish to find $\frac{4}{5}$ of $\frac{2}{3}$ of anything—and we can simplify the manipulation if we divide by 15 and multiply by 8.

With such a clear-cut understanding of what multiplication of fractions means, the class discussion can be "down to earth."



Mr. Wilson told the class, "Alec's systems help explain what we mean by

Multiplication of Fractions.

"We can better understand the meaning of:

A. $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$

if we can think of a situation in which you could find $\frac{2}{3}$ of something and then find $\frac{3}{4}$ of the result.

"Suppose we decide to think about a day (or 24 hours): $\frac{2}{3}$ of a day is 16 hours; and $\frac{3}{4}$ of this is 12 hours.

"Then we ask what fraction of a day gives us that final result.

"Alec would do the same thing:

a. $24 \left|\frac{2}{3}\right| 16 \left|\frac{3}{4}\right| 12$, $24 \left|\frac{1}{2}\right| 12$

$\left|\frac{2}{3}\right|\frac{3}{4}\left| = \right|\frac{1}{2}\left|\right.$

"How can we think about this?

B. $\frac{2}{5} \times \frac{1}{3} = \frac{2}{15}$

"We can think of June (a 30-day month) and say:

"In the last month of school, only $\frac{2}{5}$ of the days are school days, and $\frac{1}{3}$ of the school days are incomplete school days.

"We multiply 30 by $\frac{2}{5}$ to find the number of school days, which is 12 days. Then we find $\frac{1}{3}$ of that which is 4 incomplete school days. What fraction of 30 gives us the same number as the number of incomplete school days?

"Alec would write:

b. $30 \left|\frac{2}{5}\right| 12 \left|\frac{1}{3}\right| 4$, $30 \left|\frac{2}{15}\right| 4$

"How can we think about this?

C. $\frac{5}{2} \times \frac{2}{3} = \frac{5}{3}$

"At a department store, Mrs. Brown saw a piece of cloth she wanted. It was $2\frac{1}{2}$ yards long. She needed a piece that was 5 feet long. 'I wonder,' she said, 'if $\frac{2}{3}$ of that piece would be long enough.'

"First she thought of the fact that a half-yard is 18 inches. There are 5 half-yards in $2\frac{1}{2}$ yards. So, in the whole piece there are 90 inches of cloth. $\frac{2}{3}$ of that would be 60 inches, or $1\frac{2}{3}$ yards.

"Alec would write:

c. $36 \left|\frac{5}{2}\right| 90 \left|\frac{2}{3}\right| 60$, $36 \left|\frac{5}{3}\right| 60$

↑ Number of inches in the whole piece of cloth.

"How can we think about this?

D. $\frac{3}{4} \times 1\frac{1}{2} = \frac{9}{8}$ OR $1\frac{1}{8}$

"Mary had 40 books on her shelves. She said, 'I've read $\frac{3}{4}$ of them since school started.' Her sister replied, 'I've read $1\frac{1}{2}$ times as many as you have.'

"We might say that of Mary's 40 books, she had read $\frac{3}{4}$ of them, or 30 books. And $1\frac{1}{2}$ times 30 is 45. Then we want to know what fraction that is of 40 books.

"Alec might write:

d. $40 \left|\frac{3}{4}\right| 30 \left|\frac{3}{2}\right| 45$, $40 \left|\frac{9}{8}\right| 45$

"How can we think about these?

E. $\frac{1}{2} \times \frac{4}{5} = \frac{2}{5}$ and F. $\frac{3}{2} \times \frac{4}{9} = \frac{2}{3}$

+ "Make up your own problems. Show how Alec might have solved them."

e. $50 \left|\frac{1}{2}\right| 25 \left|\frac{4}{5}\right| 20$, $50 \left|\frac{2}{5}\right| 20$

f. $36 \left|\frac{3}{2}\right| 54 \left|\frac{4}{9}\right| 24$, $36 \left|\frac{2}{3}\right| 24$

+ MANY POSSIBLE ANSWERS.

Tasks: *Pupils read the story and supply the missing entries.*

Purpose: To illustrate multiplication of fractions through several applications and to define multiplication of fractions in terms of simplifying systems of multiplication and division operations.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: By this point in the discussion, pupils can be asked to suggest their own application in which one would encounter the idea of multiplying $\frac{2}{3}$ by $\frac{3}{4}$.

Here are some starters for those who need a little help in the beginning:

"It is $\frac{2}{3}$ of a mile to school."

"Mrs. Jones had $\frac{2}{3}$ of a dozen eggs."

"Mary planned to practice for $\frac{2}{3}$ of an hour."

"At the time of the fire, I saved 2 out of 3 of the books I had in my room." (This may require a statement about the number of books before the fire.)

You may wish to begin with simpler situations, such as multiplying $\frac{1}{2}$ by $\frac{1}{2}$, and supply such starters as:

"Half of the pie was left, and . . . ."

"The rocket was half-way to the moon."

"I opened a half-gallon of milk and . . . ."

"The TV show lasts half an hour and . . . ."

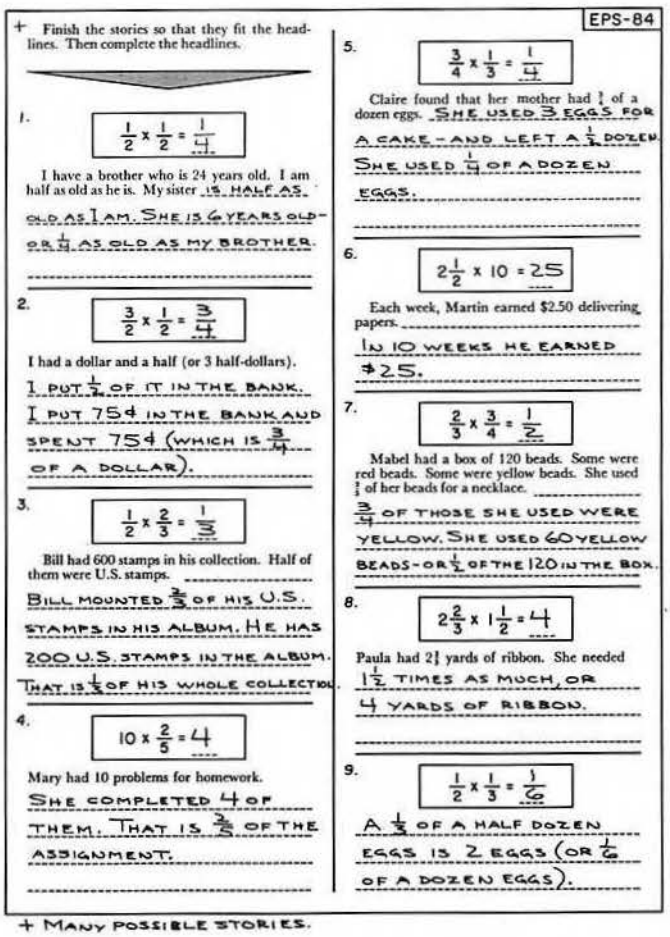
"Bill and I together earned half a dollar and . . . ."

Practice sheet 84 includes similar "starters" involving several levels of difficulty. A pupil's fluency in developing such applications is usually a reliable indication of his grasp of the whole idea of fractions. It requires much more understanding than even the most difficult manipulations carried out according to rules.

After many applications have been discussed and some written down (perhaps on the chalkboard), we are ready to consider shortcuts to the computation in each instance. We note that $\frac{2}{3}$ of a yard is 24 inches and that $\frac{3}{4}$ of 24 inches is 18 inches; but 18 inches is $\frac{1}{2}$ yard. This would have been apparent from the beginning if we had simplified the product:

$$\frac{\cancel{2}^{1}}{\cancel{3}_{1}} \times \frac{1}{\cancel{4}_{2}} = \frac{1}{2}$$

If we keep talking in terms of applications, and demonstrating that our experience in simplifying expressions is a useful tool to avoid longer computations, we have provided a persuasive motivation for the study we are making.



EPS-84

+ Finish the stories so that they fit the headlines. Then complete the headlines.

1. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

I have a brother who is 24 years old. I am half as old as he is. My sister IS HALF AS OLD AS I AM. SHE IS 6 YEARS OLD - OR $\frac{1}{4}$ AS OLD AS MY BROTHER.

2. $\frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$

I had a dollar and a half (or 3 half-dollars).
I PUT $\frac{1}{2}$ OF IT IN THE BANK.
I PUT 75¢ IN THE BANK AND SPENT 75¢ (WHICH IS $\frac{3}{4}$ OF A DOLLAR).

3. $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

Bill had 600 stamps in his collection. Half of them were U.S. stamps.
BILL MOUNTED $\frac{2}{3}$ OF HIS U.S. STAMPS IN HIS ALBUM. HE HAS 200 U.S. STAMPS IN THE ALBUM. THAT IS $\frac{1}{3}$ OF HIS WHOLE COLLECTION.

4. $10 \times \frac{2}{5} = 4$

Mary had 10 problems for homework.
SHE COMPLETED 4 OF THEM. THAT IS $\frac{2}{5}$ OF THE ASSIGNMENT.

5. $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$

Claire found that her mother had $\frac{1}{3}$ of a dozen eggs. SHE USED 3 EGGS FOR A CAKE - AND LEFT A $\frac{1}{4}$ DOZEN. SHE USED $\frac{1}{4}$ OF A DOZEN EGGS.

6. $2\frac{1}{2} \times 10 = 25$

Each week, Martin earned $2.50 delivering papers.
IN 10 WEEKS HE EARNED $25.

7. $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$

Mabel had a box of 120 beads. Some were red beads. Some were yellow beads. She used $\frac{2}{3}$ of her beads for a necklace.
$\frac{3}{4}$ OF THOSE SHE USED WERE YELLOW. SHE USED 60 YELLOW BEADS - OR $\frac{1}{2}$ OF THE 120 IN THE BOX.

8. $2\frac{2}{3} \times 1\frac{1}{2} = 4$

Paula had $2\frac{2}{3}$ yards of ribbon. She needed $1\frac{1}{2}$ TIMES AS MUCH, OR 4 YARDS OF RIBBON.

9. $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

A $\frac{1}{3}$ OF A HALF DOZEN EGGS IS 2 EGGS (OR $\frac{1}{6}$ OF A DOZEN EGGS).

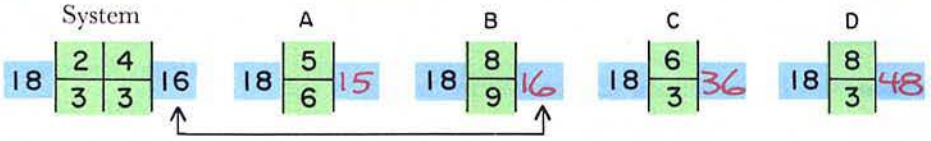
+ MANY POSSIBLE STORIES.

In the left column there are 11 questions selected from an achievement test.

Imagine you are Alec and wish to test each of the 4 possible answers given. You would select an appropriate input and try the 4-operation system and each 2-operation system. If inputs and outputs are the same, Alec would say that the systems are equal.

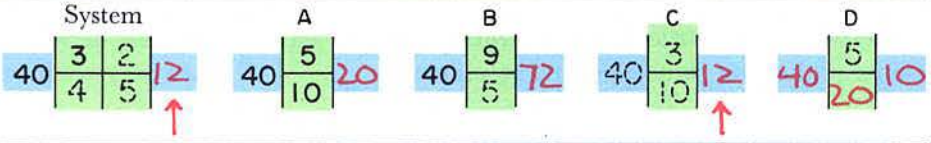
1. $\frac{2}{3} \times \frac{4}{3} = \square$ — A $\frac{5}{6}$ B $\frac{8}{9}$ C $\frac{6}{3}$ D $\frac{8}{3}$

Draw a loop around the correct answer

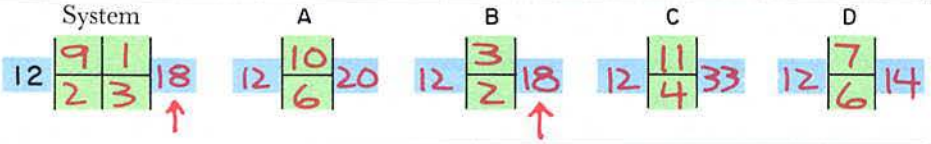


So, Alec would select B as the correct answer.

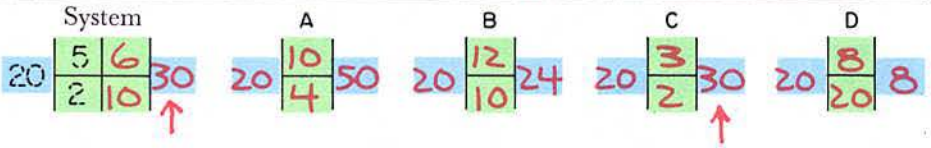
2. $\frac{3}{4} \times \frac{2}{5} = \square$ — A $\frac{5}{10}$ B $\frac{9}{5}$ C $\frac{3}{10}$ D $\frac{5}{20}$



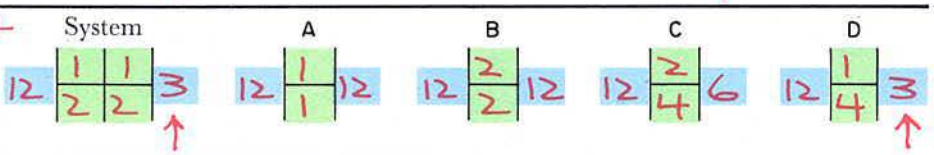
3. $\frac{9}{2} \times \frac{1}{3} = \square$ — A $\frac{10}{6}$ B $\frac{3}{2}$ C $\frac{11}{4}$ D $\frac{7}{6}$



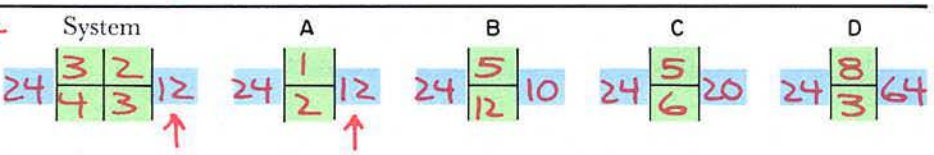
4. $2\frac{1}{2} \times \frac{6}{10} = \square$ — A $\frac{10}{4}$ B $\frac{12}{10}$ C $\frac{3}{2}$ D $\frac{8}{20}$



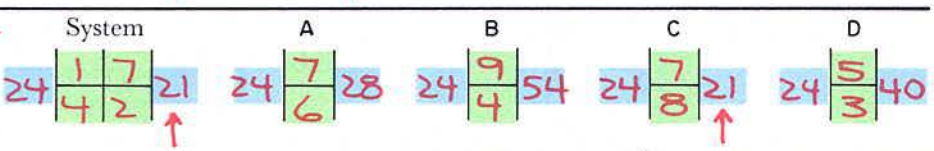
5. $\frac{1}{2} \times \frac{1}{2} = \square$ — A $\frac{1}{1}$ B $\frac{2}{2}$ C $\frac{2}{4}$ D $\frac{1}{4}$



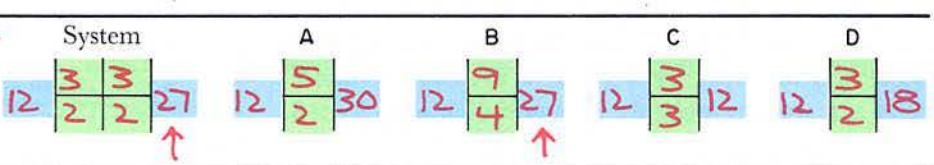
6. $\frac{3}{4} \times \frac{2}{3} = \square$ — A $\frac{1}{2}$ B $\frac{5}{12}$ C $\frac{5}{6}$ D $\frac{8}{3}$



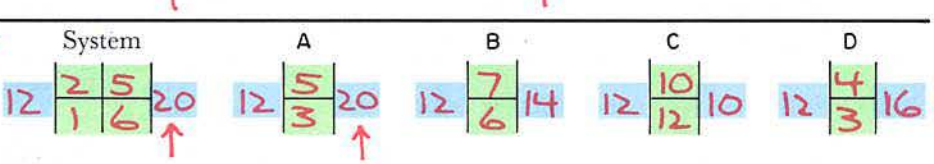
7. $\frac{1}{4} \times \frac{7}{2} = \square$ — A $\frac{7}{6}$ B $\frac{9}{4}$ C $\frac{7}{8}$ D $\frac{5}{3}$



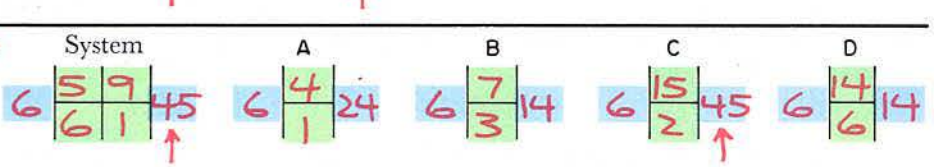
8. $1\frac{1}{2} \times 1\frac{1}{2} = \square$ — A $\frac{5}{2}$ B $\frac{9}{4}$ C $\frac{3}{3}$ D $\frac{3}{2}$



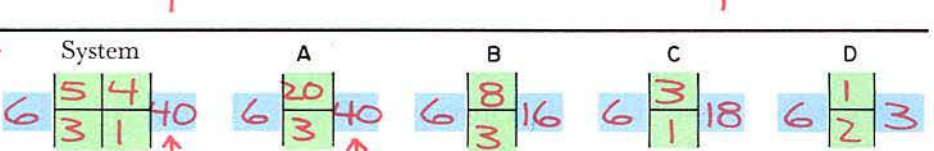
9. $2 \times \frac{5}{6} = \square$ — A $\frac{5}{3}$ B $\frac{7}{6}$ C $\frac{10}{12}$ D $\frac{4}{3}$



10. $\frac{5}{6} \times 9 = \square$ — A $\frac{4}{1}$ B $\frac{7}{3}$ C $\frac{15}{2}$ D $\frac{14}{6}$



11. $1\frac{2}{3} \times 4 = \square$ — A $\frac{20}{3}$ B $\frac{8}{3}$ C $\frac{3}{1}$ D $\frac{1}{2}$



Tasks: *Pupils solve the eleven problems in multiplying fractions and check by comparing outputs for a given input.*

Purpose: To provide practice in multiplying fractions and in checking answers.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: When pupils are confronted with questions that puzzle them because they are not sure about certain rules of manipulation, they ought to have an alternate method of attack—and Alec has.

The examples on pupil page 85 are carefully selected so that the choice of inputs is simple to make. Suppose that one of the examples had been:

$$\frac{2}{3} \times \frac{7}{8} = \begin{array}{cccc} A & B & C & D \\ \frac{5}{8} & \frac{5}{9} & \frac{11}{15} & \frac{7}{12} \end{array}$$

and suppose that Alec had selected an input of 24 thinking, perhaps, of 24 hours.

$$24 \left|\begin{array}{c|c} 2 & 7 \\ \hline 3 & 8 \end{array}\right| 14$$

Next, he tries the system A:

$$24 \left|\begin{array}{c} 5 \\ \hline 8 \end{array}\right| 15$$

He rejects that, because the outcome is not the same as with the more complicated system. So, he tries $\frac{5}{9}$:

$$24 \left|\begin{array}{c} 5 \\ \hline 9 \end{array}\right| \_\_$$

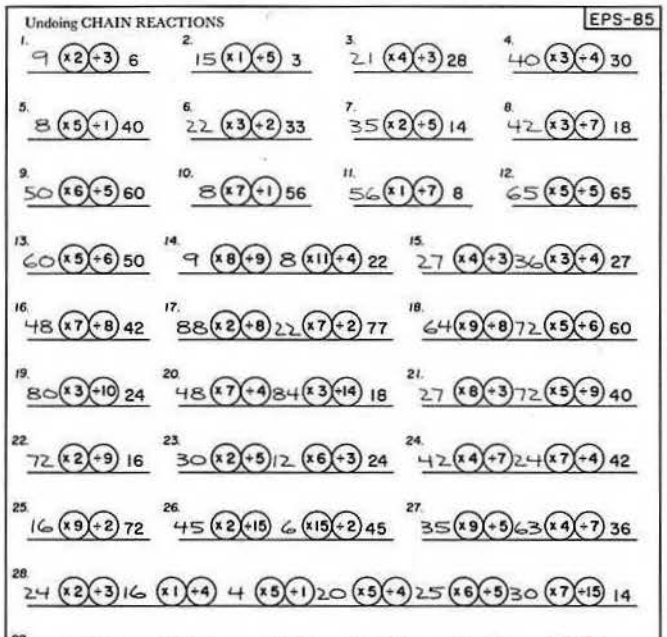
Trouble! What do you think Alec ought to do? Should he reject $\frac{5}{9}$ simply because neither 24 nor 24×5 is divisible by 9? Or, should he start over, selecting an input that is divisible by 9 as well as by 3 and 8, such as 72? If he did, he would find:

$$72 \left|\begin{array}{c|c} 2 & 7 \\ \hline 3 & 8 \end{array}\right| 42 \text{ and } 72 \left|\begin{array}{c} 5 \\ \hline 9 \end{array}\right| 40$$

so he could reject B—only to encounter trouble with $11/15$.

No, Alec is convinced that any input that will clear through a multi-operation system will clear through its 2-operation equivalent. Thus, he will reject both $\frac{5}{9}$ and $11/15$ for that simple reason. If there is any question, urge pupils to search for a counterexample—one to challenge Alec's conviction.

Practice sheet 85 anticipates the developments on pupil page 86 and practice sheet 86—undoing multiplication.



Undoing CHAIN REACTIONS — EPS-85

1. 9 (×2)(+3) 6
2. 15 (×1)(+5) 3
3. 21 (×4)(+3) 28
4. 40 (×3)(+4) 30
5. 8 (×5)(+1) 40
6. 22 (×3)(+2) 33
7. 35 (×2)(+5) 14
8. 42 (×3)(+7) 18
9. 50 (×6)(+5) 60
10. 8 (×7)(+1) 56
11. 56 (×1)(+7) 8
12. 65 (×5)(+5) 65
13. 60 (×5)(+6) 50
14. 9 (×8)(+9) 8 (×11)(+4) 22
15. 27 (×4)(+3) 36 (×3)(+4) 27
16. 48 (×7)(+8) 42
17. 88 (×2)(+8) 22 (×7)(+2) 77
18. 64 (×9)(+8) 72 (×5)(+6) 60
19. 80 (×3)(+10) 24
20. 48 (×7)(+4) 84 (×3)(+14) 18
21. 27 (×8)(+3) 72 (×5)(+9) 40
22. 72 (×2)(+9) 16
23. 30 (×2)(+5) 12 (×6)(+3) 24
24. 42 (×4)(+7) 24 (×7)(+4) 42
25. 16 (×9)(÷2) 72
26. 45 (×2)(÷15) 6 (×15)(÷2) 45
27. 35 (×9)(÷5) 63 (×4)(÷7) 36
28. 24 (×2)(÷3) 16 (×1)(÷4) 4 (×5)(÷1) 20 (×5)(÷4) 25 (×6)(÷5) 30 (×7)(÷15) 14
29. 21 (×4)(÷1) 84 (×1)(÷3) 28 (×5)(÷7) 20 (×4)(÷5) 16 (×15)(÷8) 30 (×10)(÷2) 150

a. I have saved $4.50. That is $\frac{1}{4}$ of what I need to buy a chemistry set. The set costs $6.00

b. Anne got $\frac{9}{10}$ of the answers correct. She had 4 incorrect answers. There were 40 questions and Anne got 36 correct answers.

c. Two-thirds of the number I'm thinking of is 12. The number I'm thinking of is 18

d. I'm on page 60, and that's a fourth of the way through the book. The book must have 240 pages.

e. Twenty-four students were present. That's $\frac{4}{5}$ of the class. The class must have 30 members.

f. That 30¢ magazine took $\frac{1}{3}$ of my allowance. Terry's allowance is twice as much as mine. Terry's allowance is 80¢.

g. Two-thirds of the regular price is 90¢. The regular price is $1.35

"What about

Division of Fractions?"

was Mabel's question to Mr. Wilson.

A.

"Let's remember," Mr. Wilson began, "that we can think of a division as undoing a multiplication.

"If Ben says that the model he wants costs $1.80, which is exactly 3 times as much as he has, what can you say about that situation?"

"Of course, Ben has 60 ¢."

"What actually happened was that Ben knew he had 60¢. He knew that 60¢ × 3 = $1.80. He talked about the results of a multiplication. To find out the amount he has, we need to undo the multiplication. We might say:

$1.80 (undo × 3) = 60 ¢.

"Instead, we have a name for the procedure of undoing a multiplication. We call it 'division,' and indicate it with a sign: ÷ . So, we actually write:

$1.80 ÷ 3 = 60 ¢.

÷ 3 means undo × 3

B.

"Doris said she had read 40 pages of her book, which was $\frac{2}{3}$ of all the pages.

"She may have looked at the total number of pages and found that $\frac{2}{3}$ of that was 40 pages.

60 pages × $\frac{2}{3}$ = 40 pages.

"We know only the results of the multiplication. To find the total number of pages, we simply undo the multiplication:

40 pages (undo × $\frac{2}{3}$) = 60

or, using a familiar shorthand:

40 pages ÷ $\frac{2}{3}$ = 60 pages.

÷ $\frac{2}{3}$ means undo × $\frac{2}{3}$

"If 40 pages is $\frac{2}{3}$ of the total then 20 pages, 40 ÷ 2, is one-third of the total; and 20 pages × 3 is 60 pages — the total number of pages in the book."

÷ $\frac{2}{3}$ means ÷ 2, × 3 or × $\frac{3}{2}$

C.

"Baxter had 27 correct answers. He had $\frac{9}{10}$ or 9 out of every 10 questions correct.

27 answers (undo × $\frac{9}{10}$) = 30 answers.

27 answers ÷ $\frac{9}{10}$ = 30 answers.

"If $\frac{9}{10}$ of the total is 27 then $\frac{1}{10}$ of the total is 27 ÷ 9 — or 3 answers. If that is $\frac{1}{10}$ then the total number of questions is 10 times as much: 3 pages × 10 = 30 pages

÷ $\frac{9}{10}$ means undo × $\frac{9}{10}$

÷ $\frac{9}{10}$ means ÷ 9, × 10 or × $\frac{10}{9}$

D.

"Helen paid $3.00 for a blouse. She said that that was half as much as she spent for her skirt.

"To find the cost of the skirt, we must undo × $\frac{1}{2}$. We do that by ÷ 1, × 2 or × $\frac{2}{1}$.

"Her skirt cost her $6.00."

E.

At lunch, Mr. Wilson told his family that they had already traveled $\frac{3}{4}$ of the way to the lake. They had gone 150 miles.

To find the distance from home to the lake, you must undo × $\frac{3}{4}$.

You do that by ÷ 3 × 4 or × $\frac{4}{3}$.

So, the total distance must be 200 miles.

86 Division of fractions (Undoing multiplication)

Tasks: *Pupils read the discussions and supply the missing entries.*

Purpose: To introduce the procedure which is the inverse of applying a fraction to a quantity. In other words, to introduce division of a quantity by a fraction.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: Dividing by a fraction is one of the most difficult notions in elementary school arithmetic. It is especially difficult because reasonable applications are so few. And, except in a few instances, sketches don't help much.

In the practice sheets and at many points along the way in the text, pupils have had occasion to "undo" an act of multiplication.

"The 30¢ Bill spent is ⅔ of his allowance." That means, if you take ⅔ of Bill's allowance, you will get 30¢:

$$\underline{\quad} \times \frac{2}{3} = 30$$

We have the result of a multiplication. We must "undo" that multiplication to find out the amount of Bill's allowance.

Another way we have of writing down directions to undo a multiplication is:

$$30 \div \frac{2}{3}$$

or "Starting with 30, undo a multiplication by ⅔,"

In finding ⅔ of his allowance, we would divide by 3 and multiply by 2 ($\div 3$, $\times 2$). So, to "undo" that act, we can multiply by 3 and divide by 2 ($\times 3$, $\div 2$).

Now we have two ways to give those directions:

$$30 \div \frac{2}{3} \quad \text{and} \quad 30 \times \frac{3}{2}$$

and the second form is one which we have considered at great length. So, we write:

$$30 \div \frac{2}{3} = 30 \times \frac{3}{2} = 45$$

We might, at the outset, read "$\div$ ⅔" as "undo a multiplication by ⅔," or "undo a division by 3 and a multiplication by 2."

Thus, to "invert the divisor and multiply" is a kind of summary of the way in which we undo a multiplication. Another somewhat less descriptive way is to say that we apply the "multiplicative inverse." Whatever language you prefer, we hope it means "undo a multiplication."

An analysis of the *record* of multiplication and division of whole numbers may give pupils insight into *why* we invert the divisor and multiply.

multiplication: 4 eggs x 3 = 12 eggs

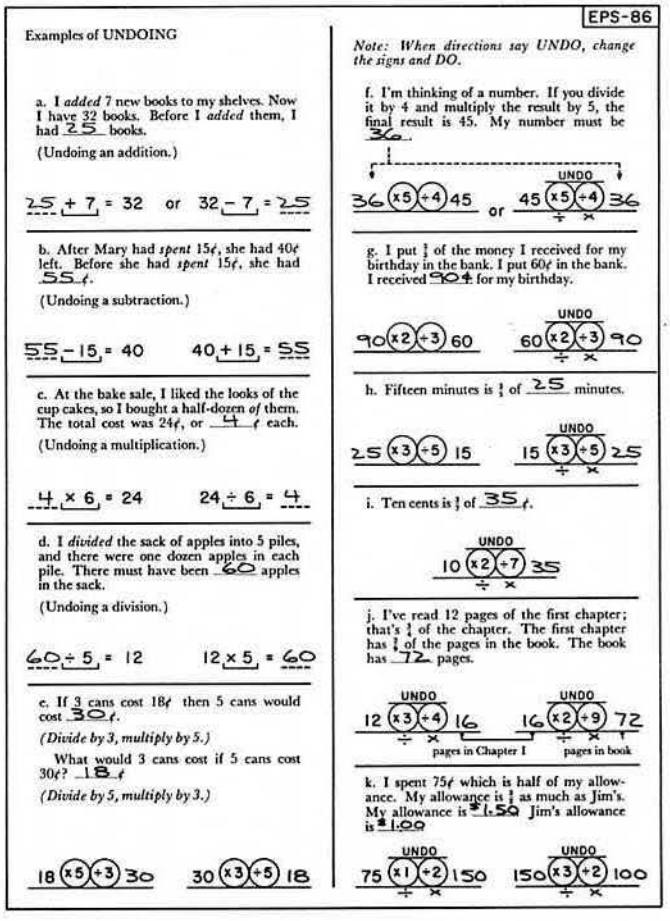
division: 12 eggs ÷ 3 = 4 eggs

But this last equation can be rewritten as:

$$12 \text{ eggs} \times \frac{1}{3} = 4 \text{ eggs}$$

Thus, dividing by 3 is the same as multiplying by ⅓.

Analyze many examples in this way with your pupils.



EPS-86

Examples of UNDOING

a. I *added* 7 new books to my shelves. Now I have 32 books. Before I *added* them, I had 25 books.
(Undoing an addition.)
25 + 7 = 32 or 32 − 7 = 25

b. After Mary had *spent* 15¢, she had 40¢ left. Before she had *spent* 15¢, she had 55¢.
(Undoing a subtraction.)
55 − 15 = 40 40 + 15 = 55

c. At the bake sale, I liked the looks of the cup cakes, so I bought a half-dozen *of* them. The total cost was 24¢, or 4¢ each.
(Undoing a multiplication.)
4 × 6 = 24 24 ÷ 6 = 4

d. I *divided* the sack of apples into 5 piles, and there were one dozen apples in each pile. There must have been 60 apples in the sack.
(Undoing a division.)
60 ÷ 5 = 12 12 × 5 = 60

e. If 3 cans cost 18¢ then 5 cans would cost 30¢.
(Divide by 3, multiply by 5.)
What would 3 cans cost if 5 cans cost 30¢? 18¢
(Divide by 5, multiply by 3.)
18 (×5) (+3) 30 30 (×3) (÷5) 18

Note: When directions say UNDO, change the signs and DO.

f. I'm thinking of a number. If you divide it by 4 and multiply the result by 5, the final result is 45. My number must be 36.
36 (×5) (÷4) 45 or 45 UNDO (×5) (÷4) 36

g. I put ¼ of the money I received for my birthday in the bank. I put 60¢ in the bank. I received 90¢ for my birthday.
90 (×2) (÷3) 60 60 UNDO (×2) (÷3) 90

h. Fifteen minutes is ¼ of 25 minutes.
25 (×3) (÷5) 15 15 UNDO (×3) (÷5) 25

i. Ten cents is ⅐ of 35¢.
10 (×2) (÷7) 35 UNDO

j. I've read 12 pages of the first chapter; that's ¼ of the chapter. The first chapter has ⅙ of the pages in the book. The book has 72 pages.
12 (×3) (÷4) 16 UNDO (pages in Chapter 1)
16 (×2) (÷9) 72 UNDO (pages in book)

k. I spent 75¢ which is half of my allowance. My allowance is ⅘ as much as Jim's. My allowance is $1.50 Jim's allowance is $1.00
75 (×1) (÷2) 150 UNDO 150 (×3) (÷2) 100 UNDO

$\div \frac{2}{3}$ means to undo $\times \frac{2}{3}$

$\times \frac{3}{2}$ will do the job.

"So, we can write: $\div \frac{2}{3} = \times \frac{3}{2}$

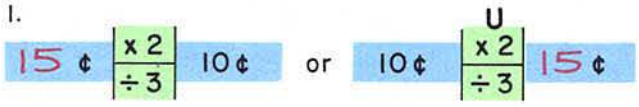
"From our work with other examples on the last page, we can write:

$\div \frac{9}{10} = \times \frac{10}{9}$ $\quad \div \frac{1}{2} = \times \frac{2}{1}$ $\quad \div \frac{3}{4} = \times \frac{4}{3}$

"Here are some of the experiments from Alec's notebook."

U means
undo the operations below.

1.



"An amount of money multiplied by $\frac{2}{3}$ is 10¢. What is the amount of money — or, undo a multiplication by $\frac{2}{3}$."

2.



3.



"All of the following examples are marked with Alec's '**U**.' To save space he did not indicate the unit of measurements in the blue blocks.

4. 20 | U 1 2 | 40

5. 30 | U 2 5 | 75

6. 18 | U 3 2 | 12

7. 36 | U 9 1 | 4

8. 45 | U 5 6 | 54

9. 28 | U 4 7 | 49

"Alec added a step to help him remember:

10. 26 | U 2 3 | ÷2 x3 | 39

11. 5 | U 1 4 | ÷1 x4 | 20

"Alec added a third step to help him remember. Then he dropped the second step.

12. 20 | U 5 8 | ÷5 x8 | 5 | 32

13. 35 | U 7 10 | ÷7 x10 | 10 7 | 50

14. 5 | U 1 8 | 8 1 | 40

15. 30 | U 3 5 | 5 3 | 50

"Alec included his **U** notation in his study of systems.

A. 40 | 3 4 | 30 | U 5 2 | 2 5 | 12 , 40 | 3 10 | 12

B. 54 | 5 6 | 45 | U 3 2 | 2 3 | 30 , 54 | 5 9 | 30

C. 20 | 7 2 | 70 | U 5 3 | 3 5 | 42 , 20 | 21 10 | 42

D. 48 | 7 8 | 42 | U 2 3 | 3 2 | 63 , 48 | 21 16 | 63

"Alec would then write:

a. $\frac{3}{4}$ U $\frac{5}{2}$ $= \frac{3}{4} \frac{2}{5} = \frac{3}{10}$

b. $\frac{5}{6}$ U $\frac{3}{2}$ $= \frac{5}{6} \frac{2}{3} = \frac{5}{9}$

c. $\frac{7}{2}$ U $\frac{5}{3}$ $= \frac{7}{2} \frac{3}{5} = \frac{21}{10}$

d. $\frac{7}{8}$ U $\frac{2}{3}$ $= \frac{7}{8} \frac{3}{2} = \frac{21}{16}$

"And we would write:

a. $\frac{3}{4} \div \frac{5}{2} = \frac{3}{4} \times \frac{2}{5} = \frac{3}{10}$

b. $\frac{5}{6} \div \frac{3}{2} = \frac{5}{6} \times \frac{2}{3} = \frac{5}{9}$

c. $\frac{7}{2} \div \frac{5}{3} = \frac{7}{2} \times \frac{3}{5} = \frac{21}{10}$

d. $\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \times \frac{3}{2} = \frac{21}{16}$

"Some people call this procedure for dividing by a fraction the *Invert and Multiply Rule.*"

87 Division of fractions (Invert and multiply rule)

Tasks: *Pupils read the discussions and supply the missing entries.*

Purpose: To introduce procedures for dividing a quantity by a fraction and for dividing a fraction by a fraction.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: Again, the approach taken enables the teacher to find applications that keep pupils thinking as they manipulate symbols.

Every question dealing with multiplication of a quantity by a fraction (that is, with applying a fraction to a quantity), such as "How much is $\frac{2}{3}$ of 15¢?," suggests a related division question. After the multiplication is caried out—$\frac{2}{3}$ of 15¢ is 10¢—we can then ask: "10¢ is $\frac{2}{3}$ of how much?"

In this case, 10¢ is the result of (1) having multiplied something by $\frac{2}{3}$, or (2) having divided something by 3 and multiplied the result by 2. To find the something, we (1) divide the final result by $\frac{2}{3}$, or (2) divide the final result by 2 and then multiply by 3. The undoing can be represented by:

$$(1)\ 10¢ \div \frac{2}{3} \quad \text{or:} \quad (2)\ 10¢ \times \frac{3}{2}$$

These expressions are two ways of directing us to undo the multiplication.

Questions involving multiplying one fraction by another (that is, for replacing a 4-operation system by a 2-operation system) can also be transformed into division questions. Here are two examples of such transformations:

Example 1. Multiplication: "Jim saved $\frac{3}{4}$ of a pie. Later, he ate $\frac{1}{3}$ of what he saved. He ate_____ of a pie."

Division: "Jim ate $\frac{1}{4}$ of a pie. That is $\frac{1}{3}$ of what he saved. He saved _____ of a pie."

Multiplication	Division
$\frac{3}{4} \times \frac{1}{3} =$	$\frac{1}{4} \div \frac{1}{3} =$

The division problem is solved by undoing the multiplication by $\frac{1}{3}$. The expressions:

$$\div \frac{1}{3} \text{ and: } \times \frac{3}{1}$$

are merely different ways of stating that a multiplication by $\frac{1}{3}$ should be undone. (Try both problems using a 24-ounce pie.)

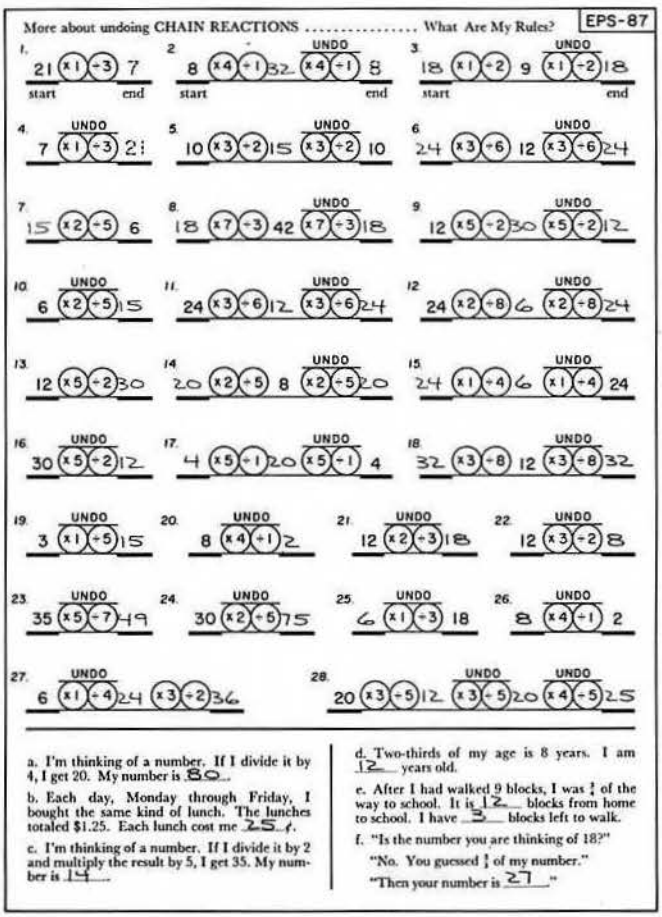
Example 2. Multiplication: "Sue had 1 pound and 5 ounces of sugar, or $21\frac{1}{16}$ of a pound. She used $\frac{2}{3}$ of what she had. She used _____ of a pound."

Division: "Sue used $\frac{7}{8}$ of a pound of sugar. That is $\frac{2}{3}$ of what she had originally. She must have started with _____ of a pound."

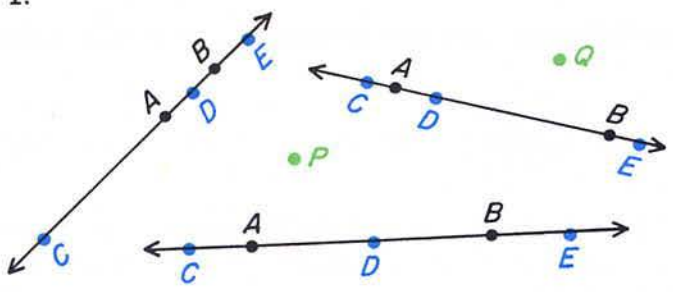
Multiplication	Division
$\frac{21}{16} \times \frac{2}{3} =$	$\frac{7}{8} \div \frac{2}{3} =$

In summary, in multiplication you (1) start with something, (2) perform a multiplication, and (3) find the result. In division you (1) start with the outcome of a multiplication, (2) "undo" the multiplication, and (3) find the subject of the original multiplication.

In this way, every application of multiplying by a fraction can be easily transformed into an application for dividing by a fraction.



I.



Lines are drawn through each of three pairs of points labeled A and B. The arrows indicate that we are thinking of straight lines (extending indefinitely).

On each line, three additional points have been labeled C, D, and E. Two other points not on the lines are labeled P and Q.

WHAT CAN YOU SAY that is true about all three of these lines?

If a statement below is *true about all three lines,* put a loop around T; if it is not true about all three, put a loop around F.

(1) The distance from C to A is less than the distance from A to D. T (F)

(2) The distance from A to C is less than the distance from C to D. (T) F

(3) D is half-way between A and B. T (F)

(4) A is between C and D. (T) F

(5) Both D and B are between C and E. (T) F

(6) The distance from E to B is less than the distance from B to D. T (F)

(7) A, B, and D are all located between C and E. (T) F

(8) D is not between A and C. (T) F

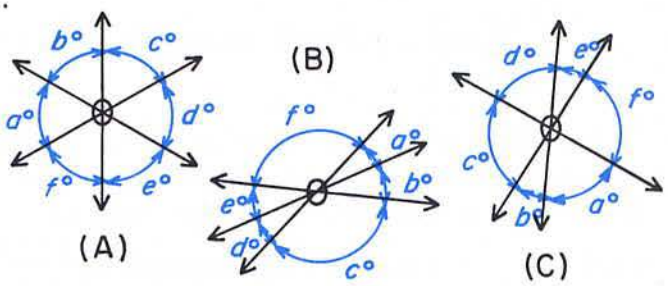
(9) If all three lines drawn were extended, each would cross the other two. (T) F

(10) If all three lines drawn were extended, a triangle would be formed. (T) F

(11) Both P and Q would be inside that triangle. T (F)

(12) A line drawn through P and Q (extended) would intersect (or cross) all three lines drawn through A and B (extended). (T) F

II.



In each sketch, three straight lines intersect at a point labeled O.

WHAT CAN YOU SAY that is true about all three sketches?

(1) $a = c$ T (F)

(2) $b + c + d = 180$ (T) F

(3) $c > a$ (c is greater than a) T (F)

(4) $c < a$ (c is less than a) T (F)

(5) $a = d$ (T) F

(6) $b + c = e + f$ (T) F

(7) $b - e = a - d$ (T) F

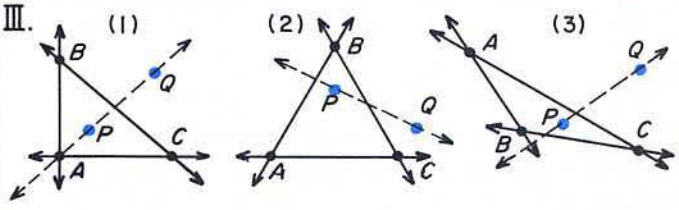
(8) $f = c$ and $e = b$ (T) F

(9) $a + b = d + e$ (T) F

(10) $a + b + c + d + e + f = 360$ (T) F

(11) $a + d = f + c$ T (F)

III.



In each sketch, the three lines intersect at points labeled A, B, and C; also P and Q are on a fourth line.

WHAT CAN YOU SAY that is true about all three sketches?

(1) The line PQ intersects the line BC. (T) F

(2) The distance between B and C is greater than the distance between A and C. T (F)

(3) The distance from A to B plus the distance from B to C is greater than the distance from A to C. (T) F

88 "What Can You Say?" (Exploring properties of certain points, lines, and angles)

Tasks: *Pupils study three sets of sketches and judge three related sets of statements.*

Purpose: To help pupils compare geometric sketches in order to discover common elements among them. To help pupils formulate *What Can You Say?* questions.

Unifying Ideas: Addition and subtraction; geometry; measurement.

The Lesson: Pupil pages 88 through 96 and many of the accompanying practice sheets utilize a tactic we highly recommend for frequent use as the basis for classroom discussion.

a. Create or describe a situation.
b. Ask pupils to think about that situation.
c. Then pose the question *What Can You Say?*

Some aspects of the situation are quite obvious, and so everyone can participate at the beginning. Other aspects are less apparent at first. Some aspects lead to relationships that are not at all the results of casual observation. Every situation worth thinking about has several levels of sophistication.

We begin on pupil page 88 with three lines. Points on the lines are labeled *A*, *B*, *C*, *D*, and *E*. (Two points not on the lines are labeled *P* and *Q*.) What can you say that is true regardless of the particular sketch you might be looking at? What is true about all three lines? We have made our own suggestions in the form of statements to be judged as true or false. Urge pupils to make their own statements—particularly statements that are interesting. For example:

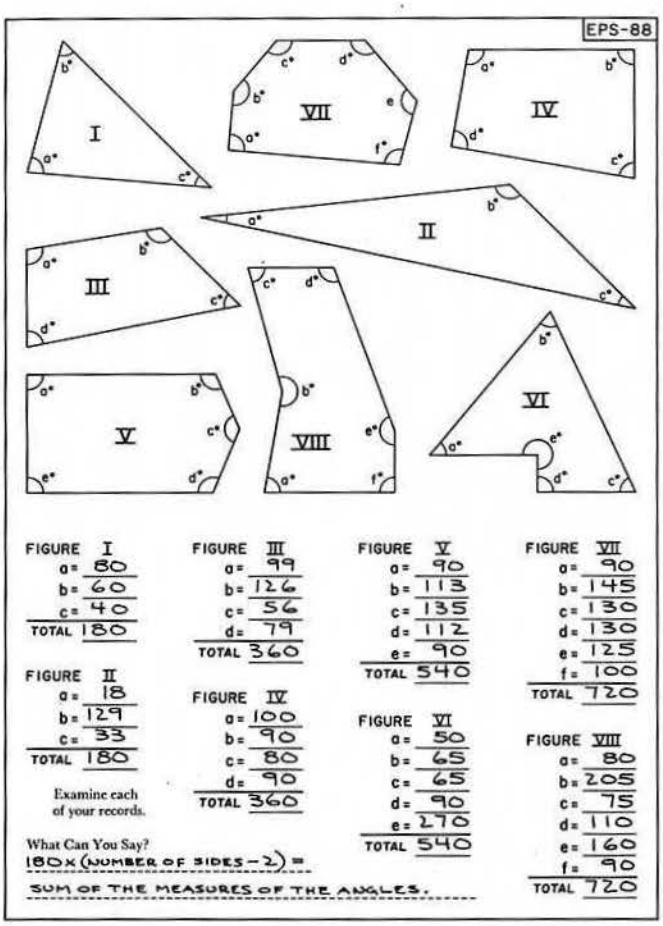
Not very interesting: There's an arrowhead at both ends of the lines drawn in the sketch.

Interesting: The five points divide the line into exactly ten parts with a labeled point at each end; the ten parts are segments *CA*, *CD*, *CB*, *CE*, *AD*, *AB*, *AE*, *DB*, *DE*, *and BE*.

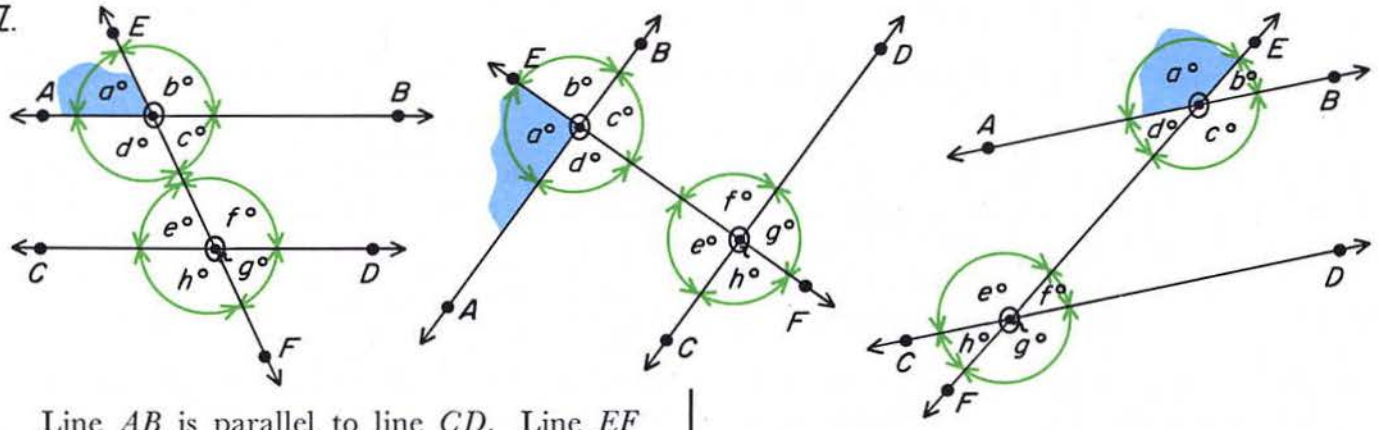
In the second set of sketches, one might point out that only two of the angles could be as large as right angles. (Someone else might point out that if there are two angles as large as right angles, they must be opposite each other.) Another way to express this idea would be to say that at least four of the labeled angles are smaller than right angles.

Some pupils might make sketches of their own and list statements, some true and some false, to challenge others in the class. Be alert for situations that lend themselves to the approach: *Here is a situation; think about it. What can you say?*

For remarks on practice sheets 88 and 89, see the Notes on pupil page 89.



Ⅳ.



Line AB is parallel to line CD. Line EF intersects line AB at O and CD at Q.

WHAT CAN YOU SAY that is true about all three sketches?

(1) O is between A and B. (T) F

(2) $d = 90$ T (F)

(3) $a = c$ (T) F

(4) $d = h$ (T) F

(5) $a + b = a + d$ (T) F

(6) $b > a$ (b is greater than a) T (F)

The angle of $a°$ formed by lines AO and EO can be referred to as $\angle AOE$ (read: angle AOE) or as $\angle EOA$.

(7) $\angle CQF \cong \angle OQD$ (T) F

(8) $b = h$ (T) F

(9) $a + b + h + g = 180$ T (F)

(10) $c + h = 180$ (T) F

(11) $\angle AOQ \cong \angle OQD$ (T) F

(12) $\angle FQD \ncong$ (is not congruent to) $\angle AOE$ T (F)

List some of your own statements that are true about all 3 sketches above.

CD IS NOT PARALLEL TO EF.

c+d+e+f = 360.

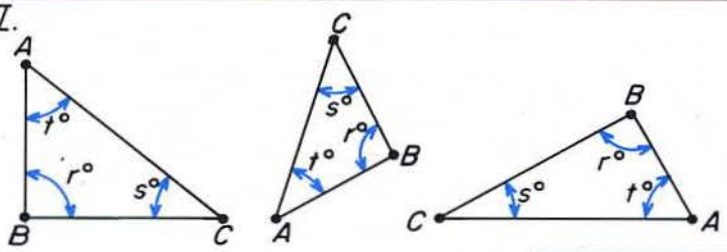
a+b+c-d = g+e+f-h.

a IS NOT SMALLER THAN g.

a+b-c = b

a+g = c+e

Ⅴ.



The line segments AB and BC are perpendicular — they form a right angle.

WHAT CAN YOU SAY about one of the triangles that is true for all of them?

(1) $r = 90$ (T) F

(2) $\angle ABC$ is a right angle. (T) F

(3) The length of AC is greater than the length of BC. (T) F

(4) $r + s + t = 180$ (T) F

(5) $r + s > 130$ T (F)

(6) $s + t = 90$ (T) F

(7) $r - s = t$ (T) F

(8) $r > t$ (T) F

(9) $s > t$ T (F)

(10) $r > s$ (T) F

(11) $s = t$ T (F)

List some of your own statements that are true for all 3 sketches.

r-(s+t) = 0

AB+BC IS LONGER THAN AC.

AC IS LONGER THAN AB

2r+2s+2t = 360

s ≠ t r-t = s

∠BAC IS NOT A RIGHT ANGLE.

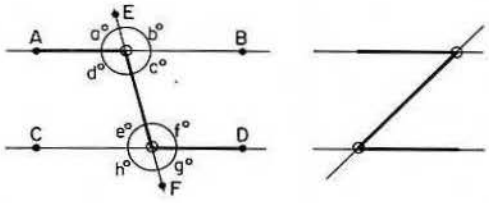
89 Exploring congruent and noncongruent angles

Tasks: *Pupils study two sets of figures and judge two related sets of statements. They make their own true statements about all the figures in each set.*

Purpose: To help pupils see significant relationships in sets of geometric sketches. To extend study of the Z-result and right triangles.

Unifying Ideas: Structure; functions and relations; geometry; measurement.

The Lesson: The first set of sketches extends our study of the Z-result on pages 54 and 55.



The key ideas here are:

1. $d=f$ and $c=e$ (the Z-result)
2. $a=c$ and $b=d$
3. $d+e=180$
4. $a=e$ and $d=h$
5. $a+b=180$

These ideas will arise often. They are intuitively obvious and ought to be treated as such. They are based on such simple arguments as:

Since	$a=c$
and	$c=e$
it follows that	$a=e$.

Remember that at this level we are not interested in explaining what constitutes an acceptable proof in geometry.

Encourage pupils to make interesting if–then statements, such as: "If $a=b$ then line EF is perpendicular to lines AB and CD" and "If a line were drawn connecting points B and C then it would cross line EF between O and Q, forming two triangles (and each angle of one triangle would be the same size as an angle in the other)."

The second set of sketches consists of three right triangles with $\angle ABC$ the right angle. Useful ideas include:

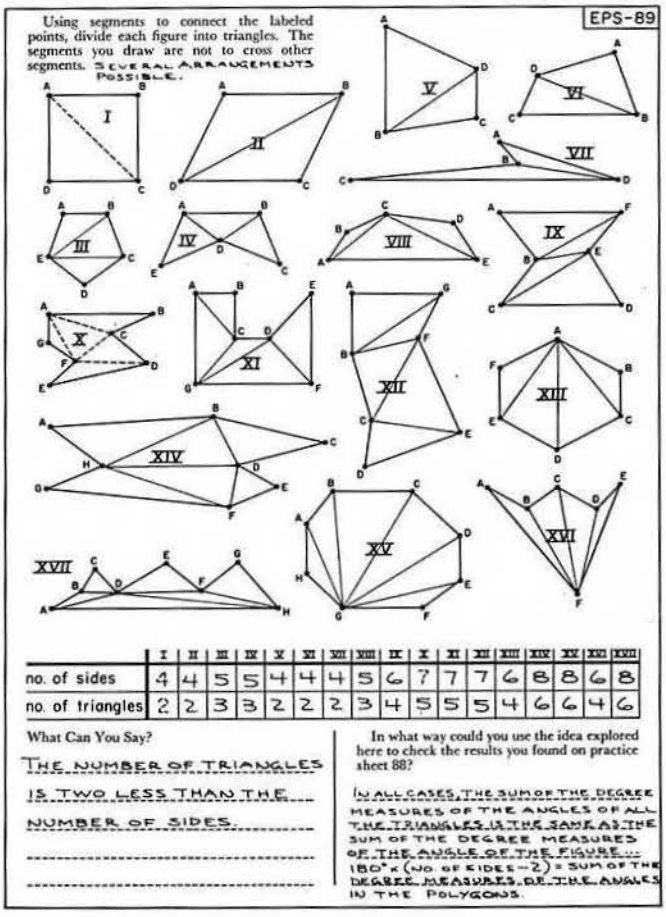
1. AC is the longest side. It is opposite the right angle. (You may wish to introduce the word *hypotenuse* for this side.)
2. $s+t=90$
3. If sides AB and BC are the same length then $s=t=45$.

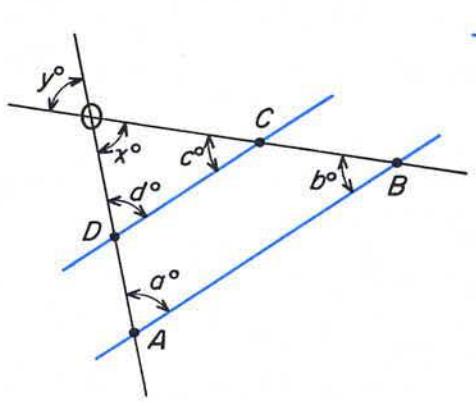
(The sign "$\cong$" is read as "is congruent to." Two angles are congruent if and only if they are the same size.)

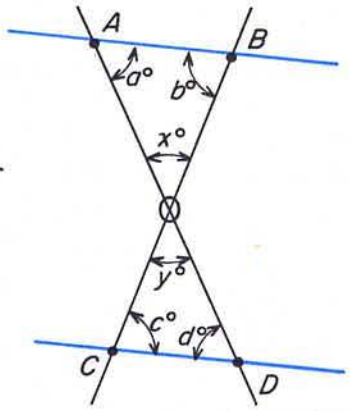
Practice sheet 89 takes a different approach from that on practice sheet 88 to developing a formula for the sum of the angle-measures of a polygon. Any polygon can be partitioned into triangles by connecting vertices with nonintersecting lines. The number of triangles will be two less than the number of sides. The sum of the angle-measures of the triangles will be the sum of the angle-measures of the polygon. Since the sum of the angle-measures of a triangle is 180°, we have the formula:

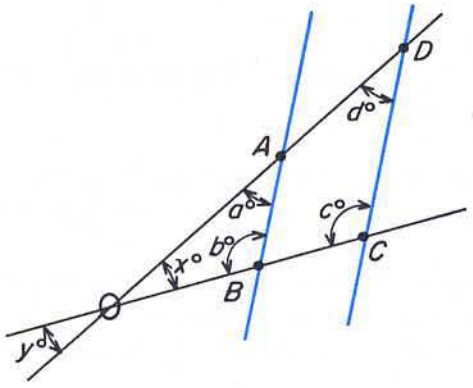
180° × (number of sides–2) = the sum of the angle-measures of a polygon.

(Of course, this assumes that the word *angle* includes *reflex angles*, angles with degree-measures greater than 180.)


EPS-89
Using segments to connect the labeled points, divide each figure into triangles. The segments you draw are not to cross other segments. SEVERAL ARRANGEMENTS POSSIBLE.
I
II
III
IV
V
VI
VII
VIII
IX
X
XI
XII
XIII
XIV
XV
XVI
XVII
no. of sides 4 4 5 5 4 4 4 5 6 7 7 7 6 8 8 6 8
no. of triangles 2 2 3 3 2 2 2 3 4 5 5 5 4 6 6 4 6
What Can You Say?
THE NUMBER OF TRIANGLES IS TWO LESS THAN THE NUMBER OF SIDES.
In what way could you use the idea explored here to check the results you found on practice sheet 88?
IN ALL CASES, THE SUM OF THE DEGREE MEASURES OF THE ANGLES OF ALL THE TRIANGLES IS THE SAME AS THE SUM OF THE DEGREE MEASURES OF THE ANGLE OF THE FIGURE.
180°×(NO. OF SIDES−2) = SUM OF THE DEGREE MEASURES OF THE ANGLES IN THE POLYGONS.







I. In each sketch we have drawn a pair of parallel lines. We have drawn two more lines that intersect each other at O. One of these two lines intersects the parallel lines at A and D; the other intersects them at B and C.

WHAT CAN WE SAY that applies to all of the sketches?

If the angles of one triangle are the same size as the angles of another triangle, we say the triangles are SIMILAR — they have the same shape. A backward "S" on its side is shorthand for the idea.

"$\sim$" means *is similar to*

(1) $a = d$ (T) F

(2) $d = b$ T (F)

(3) $x = c$ T (F)

(4) $b = c$ (T) F

(5) $x = y$ (T) F

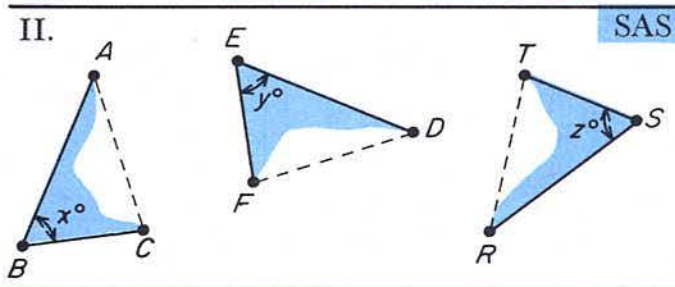
(6) $a + b = c + d$ (T) F

(7) $\triangle ODC \sim \triangle OAB$ (T) F

(8) $x + d + c = x + a + b$ (T) F

(9) $\triangle BOA$ is the same size and shape as $\triangle COD$. T (F)

II. SAS

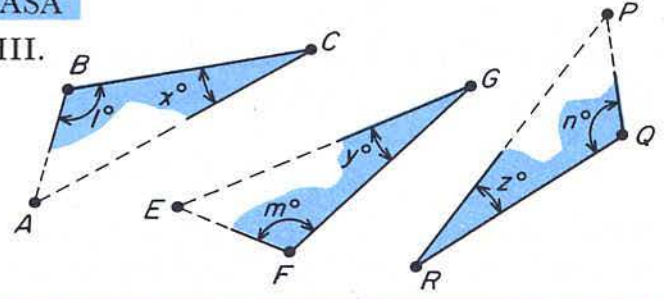


IF sides AB, DE, and RS are the same length; and if sides BC, EF, and ST are the same length; and if $x = y = z$,

THEN What Can You Say?

+ THE TRIANGLES ARE THE SAME SIZE AND SHAPE.

III. ASA

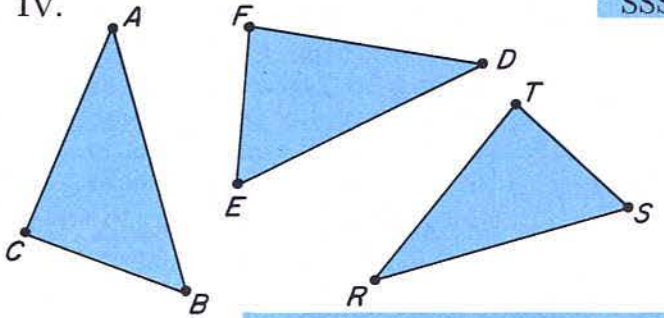


IF sides BC, FG, and RQ are the same length, and if $l = m = n$ and $x = y = z$,

THEN What Can You Say?

+ THE TRIANGLES ARE THE SAME SIZE AND SHAPE.

IV. SSS



IF sides AB, DE, and RS are the same length; and if sides BC, EF, and ST are the same length; and if sides AC, DF, and RT are the same length,

THEN What Can You Say?

+ THE TRIANGLES ARE THE SAME SIZE AND SHAPE.

"$\cong$" means *is congruent to* or *is the same size and shape as*

+ MORE CAN BE SAID.

90 Exploring similar triangles, congruent triangles (SAS, ASA, SSS)

Tasks: *Pupils study four sets of geometric sketches. In the first set they judge mathematical statements about each of the sketches. In the other three sets they formulate a response to* What Can You Say? *situations based on a group of given conditions.*

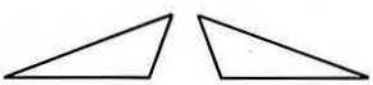
Purpose: To help pupils observe significant relationships in geometric figures. To extend study of the Z-result and to introduce the geometric ideas of similarity and congruence of triangles.

Unifying Ideas: Functions and relations; geometry; measurement.

The Lesson: The ideas developed in the first sequence of sketches (*IV*) on pupil page 89 are used in finding significant relationships in the three sketches at the top of pupil page 90. In each sketch, two triangles are formed—OAB and OCD. The most important relationship is that $a = d$, $b = c$, and $\angle AOB$ is the same size as $\angle DOC$. So, in each pair of triangles, the angles of one are the same size as the angles of the other.

When this is true of two triangles, we call them *similar* and often use the symbol "$\sim$" as an abbreviation for *is similar to*. We sometimes say that such triangles have the same shape. Of course, triangles can have the same shape without being the same size. How many triangles of different sizes could be drawn that are the same shape as a given triangle? The answer is "as many as you wish, both larger and smaller."

But triangles may have both the same shape and the same size—one could be cut out and fitted nicely on the other, even though it might have to be turned over, as with these two:

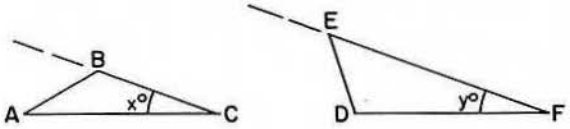


When two triangles have the same shape and size, we say that they are *congruent* and use "$\cong$" as an abbreviation for *is congruent to*.

The three sequences of sketches on the lower half of pupil page 90 deal with congruent triangles. Each series considers one of the three general situations when enough information is given to know that the triangles are the same shape and size:

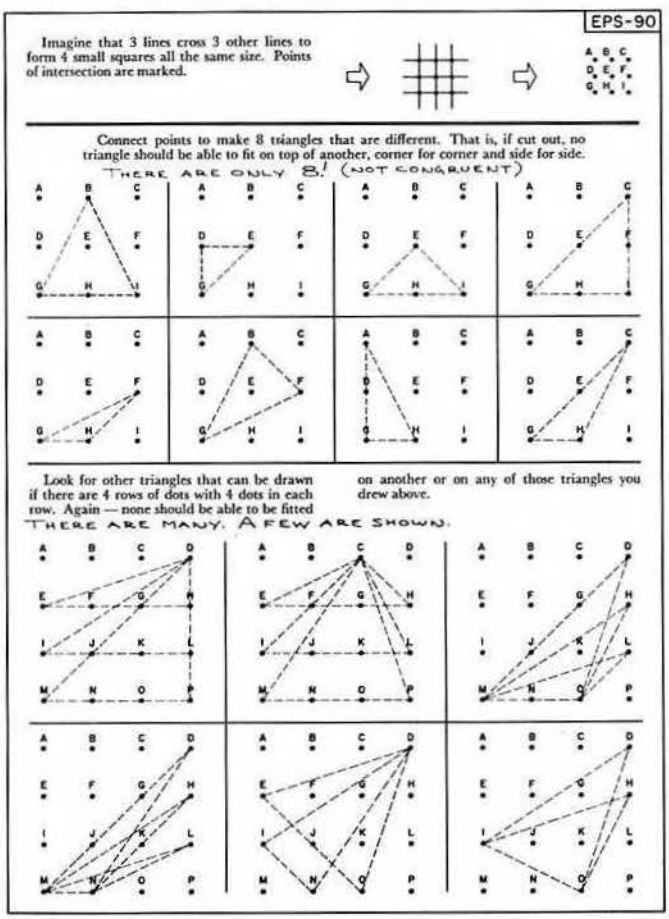
SAS . . . side-angle-side
ASA . . . angle-side-angle
SSS . . . side-side-side

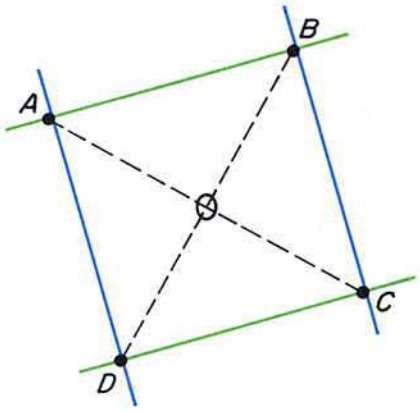
Question: If two sides and one of the angles not between the sides are the same in two triangles, are the triangles congruent? At first, it might seem that any two such triangles would be congruent, but notice the following exception:

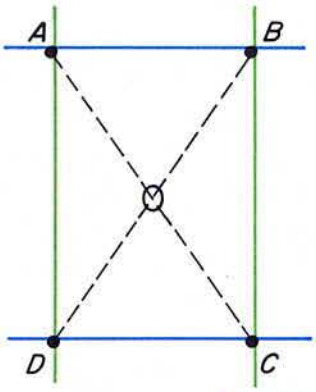


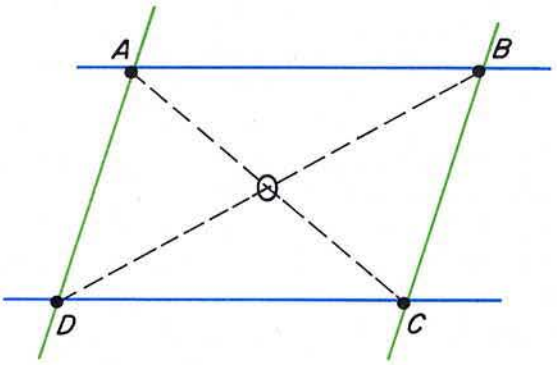
Here, sides AB and DE are the same length and so are sides AC and DF; also, $x = y$. But, clearly, the triangles ABC and DFE are not congruent.

Practice sheet 90 will help develop a deeper understanding of the idea of congruence.









A pair of parallel lines (blue) intersects another pair of parallel lines (green). The points of intersection are labeled *A, B, C,* and *D*. Diagonals have been drawn connecting *A* and *C,* and connecting *B* and *D*. The intersection of the diagonals is labeled *O*. ---------->

Note: The 4-sided figure created when two pairs of parallel lines intersect — such as the figures we have labeled *ABCD* — are called PARALLELOGRAMS.

All squares and rectangles are usually considered as special cases of parallelograms.

What Can You Say?

(1) *ABCD* is a parallelogram. (T) F

(2) *ABCD* is a square. T (F)

(3) *ABCD* is a rectangle. T (F)

(4) The distance between *A* and *B* is the same as the distance between *B* and *C*. T (F)

(5) The distance between *A* and *D* is the same as the distance between *B* and *C*. (T) F

(6) The diagonal *AC* has the same length as the diagonal *BD*. T (F)

(7) The distance between *A* and *O* is the same as the distance between *O* and *C*. (T) F

(8) ∠ *ABC* is a right angle. T (F)

(9) ∠ *DOC* is the same size as ∠ *DOA*. T (F)

(10) △ *ABC* is a right triangle. T (F)

(11) △ *ABC* is the same size and shape as △ *CDA*. (T) F
(If 2 figures have the same size and shape, one could be cut out and put down on the other with all the points of one fitting the points of the other exactly.)

(12) △ *ABD* is the same size and shape as △ *BCA*. (T) F

(13) ∠ *BAO* is the same size as ∠ *DCO*. (T) F

(14) ∠ *ADC* is the same size as ∠ *ABC*. (T) F

(15) The distance between *D* and *O* is the same as the distance between *O* and *B*. (T) F

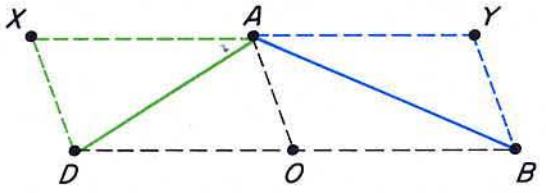
(16) △ *ABO* is the same size and shape as △ *CDO*. (T) F

(17) ∠ *BOC* is the same size as ∠ *AOB*. T (F)

(18) △ *AOD* is the same size and shape as △ *COB*. (T) F

(19) △ *AOB* is not the same shape as △ *AOD*, but it is the same size. (T) F

Consider statement (19) further. Let's redraw those triangles from the sketch on the right:



We draw *DX* and *YB* parallel to *AO* and *XA* and *AY* parallel to *DB*. Remember that the sides *DO* and *OB* have the same length (15).

(20) *AODX* and *YBOA* are parallelograms of the same size and shape. (T) F

(21) △ *ADO* and △ *DXA* have the same size and shape. (T) F

91 "What Can You Say?" (Exploring properties of parallelograms)

Tasks: *Pupils study the three sketches and judge statements related to each of the sketches.*

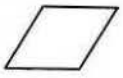
Purpose: To help pupils compare geometric figures to discover common elements among them. To help pupils formulate *What Can You Say?* questions.

Unifying Ideas: Geometry; measurement.

The Lesson: A square is a special case of a parallelogram. What can you say about a square?

All four sides are equal in length.
All four angles are right angles.
The diagonals are equal in length.
The diagonals divide the square into four congruent triangles—right triangles and isoceles triangles (called "right isoceles triangles").

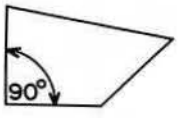
If the four sides of a quadrilateral are equal in length, can you be sure that it is a square? No:



If the four angles are right angles, can you be sure that the quadrilateral is a square? No:

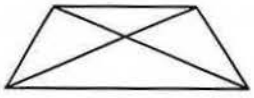


If four sides are equal in length and four angles are right angles, is it a square? Yes. In fact, if four sides are equal in length and one angle is a right angle, it is a square. Is it true that if three sides are equal in length and one angle is a right angle, it is a square? No:



What about three sides equal in length and two right angles? (It is a square.) What about two sides equal in length and three right angles? (It may not be a square, but it must be a rectangle.)

If its diagonals are equal in length, must it be a square? A rectangle? (No, to both questions.)



If its diagonals have the same length and bisect each other, is it a square? Is it a rectangle? (It must be a rectangle, but need not be a square.)

If the diagonals of a quadrilateral bisect each other, is it a rectangle? (It must be a parallelogram, but is not necessarily a rectangle.)

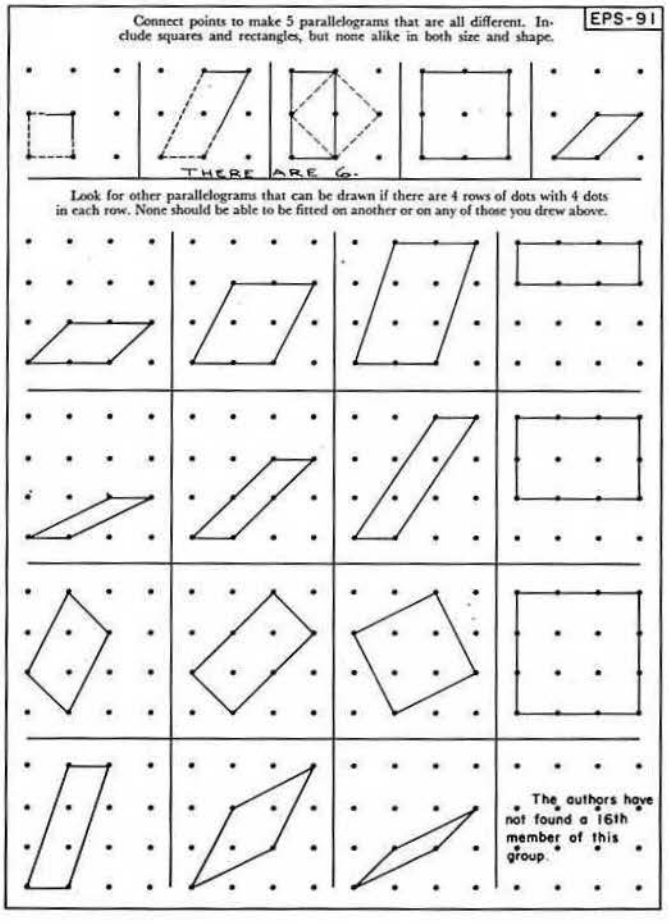
If one pair of opposite sides of a quadrilateral are parallel and equal in length, is it a parallelogram? (Yes.)

Consideration of such questions sometimes leads to heated discussions and always to a deeper understanding of the properties of these basic geometric shapes.

You might introduce another member of this family—a *trapezoid* (a quadrilateral with only one pair of parallel sides).



Someone might like to extend the investigation carried out on practice sheet 91 by using a grid of dots arranged in a 5-by-5 array.



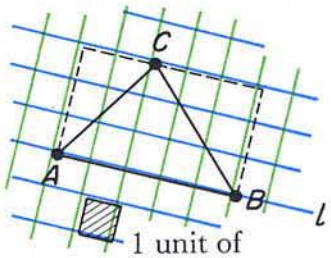
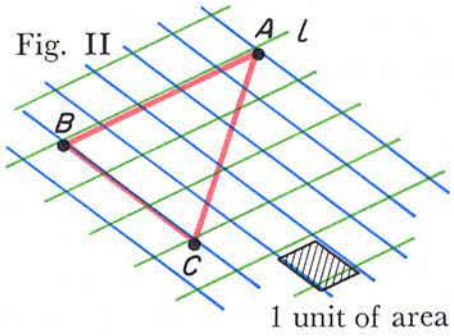
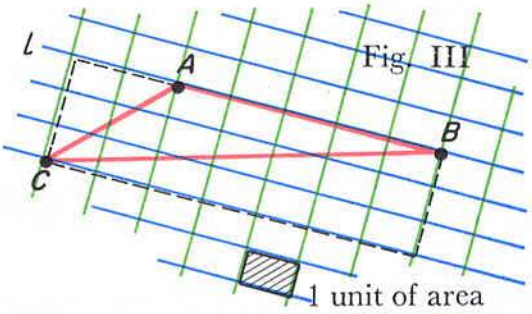
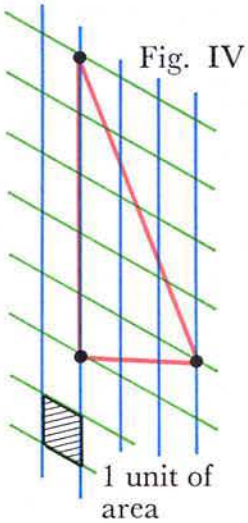
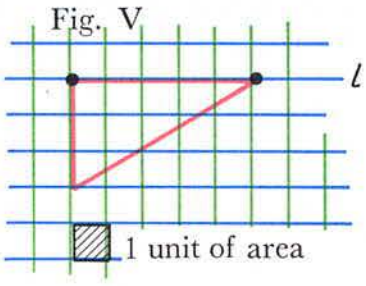


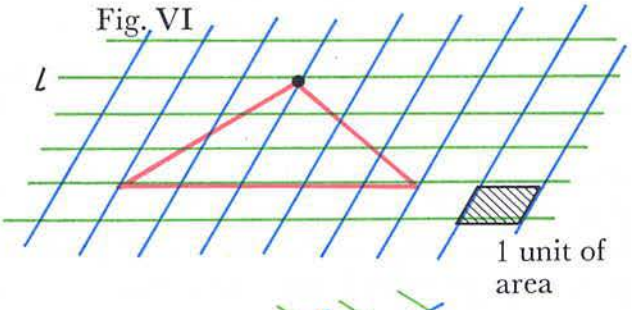
Fig. I

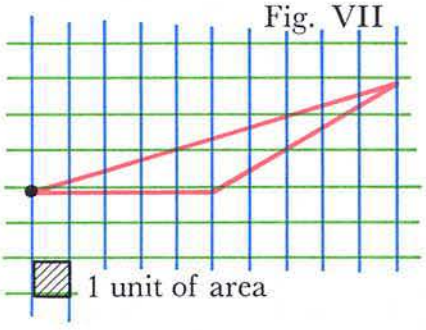


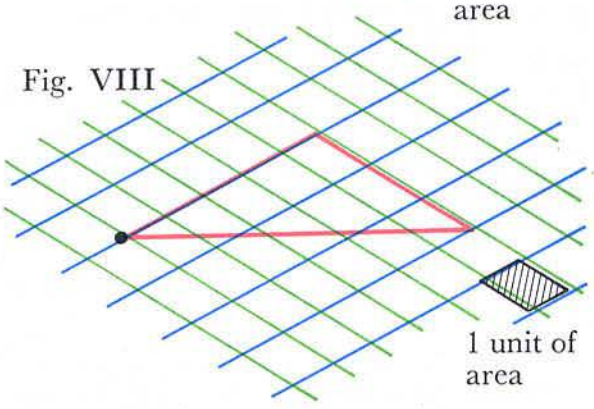












A set of equally spaced parallel lines is drawn to intersect another set of equally spaced parallel lines.

A line is selected and labeled l.

On l, a point A is located where a line intersects l. Also on l, a point B is located 5 spaces from point A.

A point C is located at any intersection on a line parallel to l and 3 spaces from l.

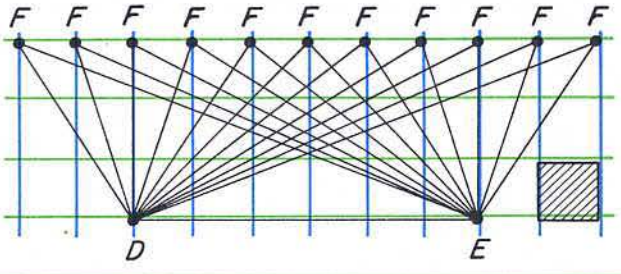
Lines are drawn connecting points A and B, points B and C, and points A and C.

(Only Figure I is completed according to the description above. Please complete the other figures by following the description.)

(Additional broken lines are drawn in Figures I and III that may help you with the next question.)

What can you say about $\triangle ABC$ that is true with regard to all the figures?

EACH TRIANGLE HAS AN AREA THE SIZE OF $7\frac{1}{2}$ OF THE UNITS OF AREA IN THAT FIGURE.



What can you say about the area of triangles labeled DEF?

THE AREA OF EACH IS 9 UNITS.

92 Exploring the concept of area of a triangle

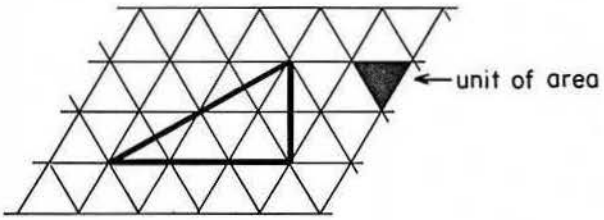
Tasks: *Pupils read and follow directions for constructing triangles to be labeled* ABC *on seven grids and then try to formulate a statement that would be true of each such triangle. Further, pupils try to formulate a statement that would be true of all triangles labeled* DEF *in another construction.*

Purpose: To provide practice in following directions for geometric constructions. To help pupils compare geometric figures to discover common elements among them.

Unifying Ideas: Geometry; measurement.

The Lesson: When we talk about finding areas of figures, we define a unit of area and then find the number of those units required to cover the region bounded by the figure. (It is common practice to talk of the area of a triangle or rectangle when what is meant is the area of a triangular or rectangular *region.* We shall use the more precise way of speaking in just these comments.)

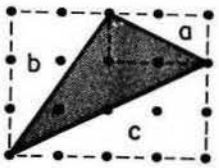
Because we customarily discuss area in terms of square units, many get the notion that we must have squares in order to define an area unit. Pupil page 92 is designed to indicate that other shapes can be used in selecting a unit. You may wish to go still further to indicate that triangular units could be used.



Or, some pupils might like to explore this idea on their own.

As an extension of the idea introduced on pupil page 92 and in anticipation of the work to come on pages 123 through 128, pupils can draw their own shapes on graph paper and determine the number of units in each.

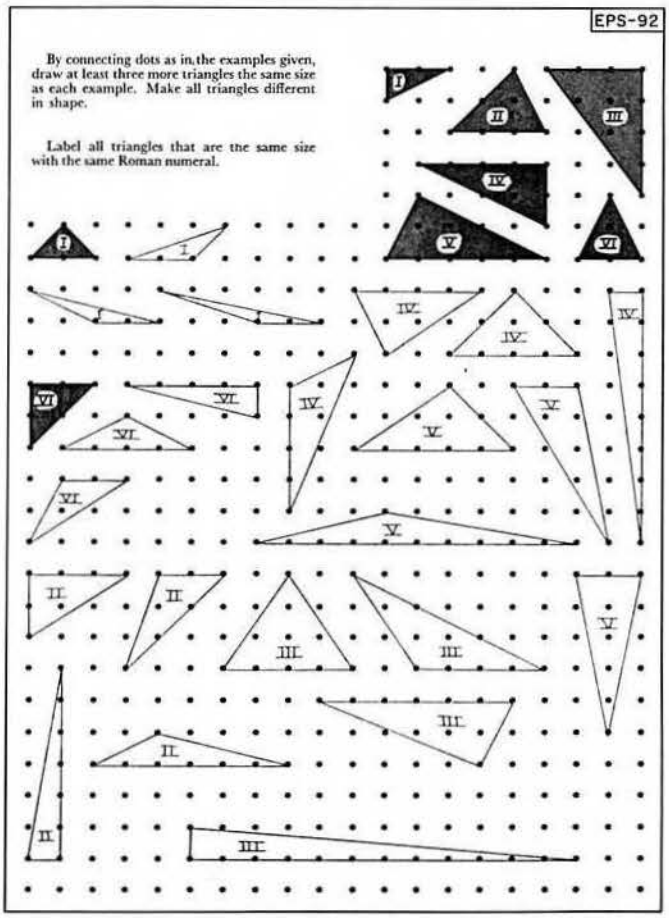
Every triangular region can be considered part of a rectangular region.



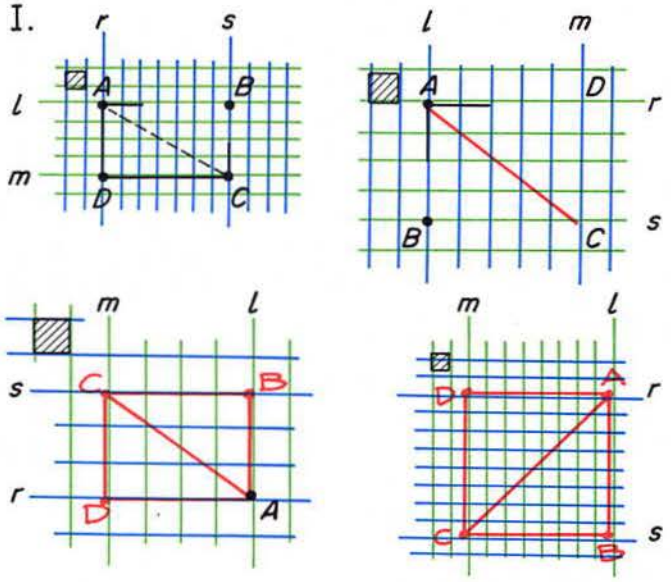
Each part of the rectangular region that is not part of the triangular region can be seen as a triangular region that is half the size of a smaller rectangular region. Since the areas of rectangular regions are easily found, it follows that the area of half a rectangular region is easily found. The sum of their areas is 8 square units. Since the area of the large rectangular region is 3 × 4, or 12, square units, the shaded area must be 12–8, or 4, square units.

Read the Notes on Pupil Page 124 for a further description of this method of determining the area of a triangular region.

Similar methods can be used to find the area of any region made by connecting intersections of the lines on graph paper—regions bounded by any desired number of sides. Perhaps some pupils would like to challenge others to find the areas of rather intricately shaped regions. Or, someone might think he can draw a region whose area can't be determined —which is quite a challenge.



I.



Follow the directions and complete the sketches.

1. One set of equally spaced parallel lines intersects another set with the same equal spacing at right angles.

2. Two lines in one set are selected and labeled r and s. Two lines in the other set are selected and labeled l and m.

3. The intersection of l and r is labeled A, of l and s is labeled B, of s and m is labeled C, of r and m is labeled D. A unit of area is indicated.

4. Draw sides AB, BC, CD, and DA and diagonal AC.

What can you say that is true about all the sketches?

(1) $ABCD$ is a rectangle. (T) F

(2) $ABCD$ is a parallelogram. (T) F

(3) The side AB is longer than the side BC. T (F)

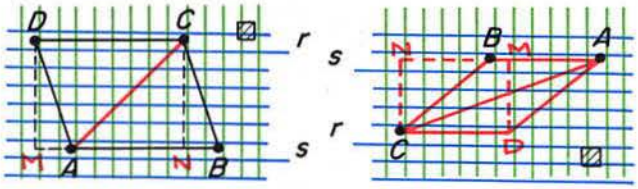
(4) There are as many equal spaces between A and B as between C and D. (T) F

(5) The number of units of area of rectangle $ABCD$ is the same as the number of spaces between A and B multiplied by the number of spaces between A and D. (T) F

(6) The number of units of area in rectangle $ABCD$ is twice the number of units of area in triangle ABC and twice as many as in triangle ADC. (T) F

(7) $\triangle ABC \cong \triangle CDA$ (T) F

II.



Follow the directions and complete the sketches.

1. Two sets of parallel lines with the same equal spacing intersect.

2. Points A and B are located on line s. On a line parallel to s and labeled r, two points, D and C, are located the same number of equal spacings apart as A and B — labeled so that lines AD, DC, CB, and BA form a parallelogram.

3. Draw line AC. Draw broken lines from D and C perpendicular to line s. Label these lines DM and CN. Connect M and A and connect N and B (unless they are already connected).

What can you say about both sketches?

(1) $\triangle ABC \cong \triangle CDA$ (T) F

(2) $\triangle DMA \cong \triangle CNB$ (T) F

(3) The distance between B and M is greater than the distance between A and N. T (F)

(4) $ABCD$ is a rectangle. T (F)

(5) $DCNM$ is both a rectangle and a parallelogram. (T) F

(6) The number of area units in rectangle $MNCD$ is the number of spaces between points A and B multiplied by the number of spaces between lines r and s. (T) F

(7) The number of area units in $\triangle DMA$ is the same as the number of area units in $\triangle CNB$. (T) F

(8) The area of $\triangle ADC$ equals the area of $\triangle ACB$. (T) F

(9) The area of $\triangle ACB$ is the sum of the areas of $\triangle ACN$ and $\triangle CNB$. (T) F

(10) The area of $\triangle ADC$ is the sum of the areas of $\triangle ACN$ and $\triangle DMA$. (T) F

(11) The area of $\triangle ADC$ is half the area of parallelogram $ABCD$ and is also half the area of rectangle $CDMN$. (T) F

(12) $DCNA$ is a parallelogram. T (F)

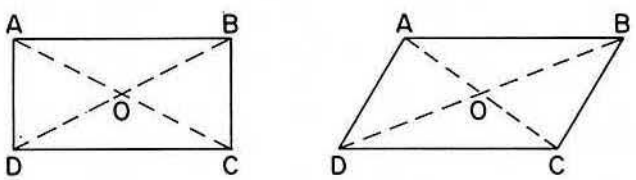
93 Exploring the concept of area of a rectangle

Tasks: *Pupils follow directions to complete two sets of sketches and then judge statements about the sketches in each set.*

Purpose: To help pupils observe significant relationships in sets of geometrical sketches. To extend the study of the geometric ideas of area and congruence.

Unifying Ideas: Functions and relations; geometry; measurement.

The Lesson: After the pupils have considered the sketches in both discussions on pupil page 93, draw the other diagonal in each case and ask what else can now be said that is true about both.



The point at which the diagonals intersect is labeled *O*.

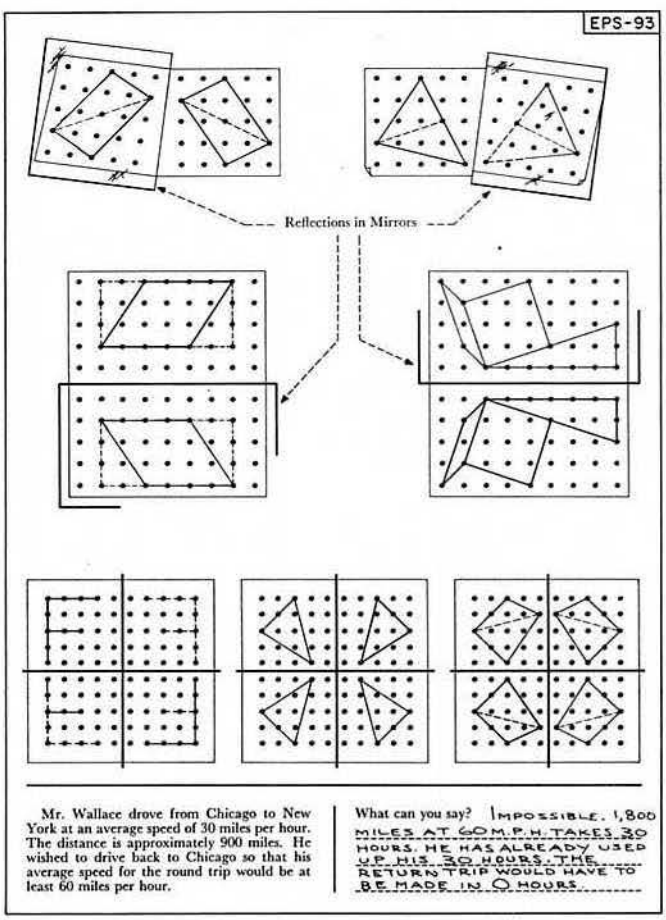
In each case, the diagonals have formed four small triangles: *ABO*, *BCO*, *CDO*, and *ADO;* and four larger triangles: *ABC*, *BCD*, *ACD*, and *ABD*.

Let's consider the areas of these triangles. The work on pupil page 93 leads us to say that in each case the large triangles have the same area—each is ½ the area of the rectangle or parallelogram. And, it doesn't take much argument to show that opposite triangles *(AOB* and *COD; AOD* and *BOC)* are congruent—having both the same size and same shape.

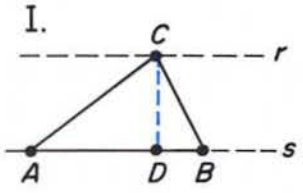
But what can we say in comparing the areas of triangles *AOD* and *AOB?* Clearly, they are not the same shape. Might they, however, be the same size—have the same area?

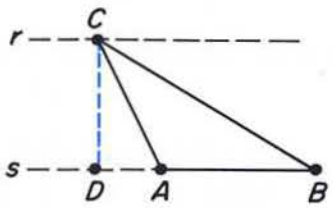
Let pupils use all their powers of persuasion to show that they must be the same size or that they need not be the same size. Have them draw examples on graph paper and compute the areas. (They are, of course, the same size. However, we hope pupils will be given full opportunity to argue both sides of the question and convince themselves without recourse to outside authority.)

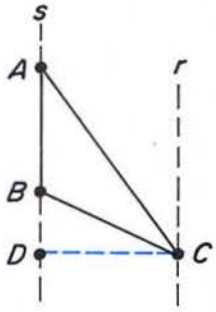
In geometry, the ability to visualize is a handy tool for most pupils. It is particularly helpful if one has some experiences in seeing the same shape in different positions. The notion of congruence requires this. Practice sheets 93 and 94 provide some experience with this idea. A piece of graph paper is all that is needed.

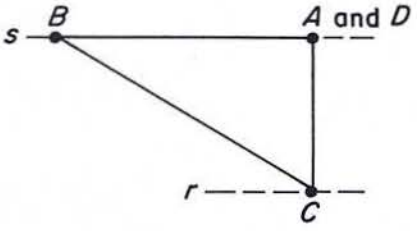


I.









Two sets of equally spaced parallel lines intersect at right angles. From one set, lines r and s are selected. Points A and B are located at intersections on s. Point C is located at an intersection on line r. A line from C is drawn perpendicular to s. It meets s at a point labeled D. (In special cases, points A and D may be the same.)

Draw triangle ABC.

What can you say about the area of $\triangle ABC$ in all of the sketches?

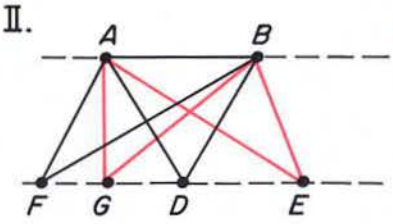
The number of spaces between r and s (or C and D) multiplied by the number of spaces between A and B is twice the number of area units in $\triangle ABC$. (T) F

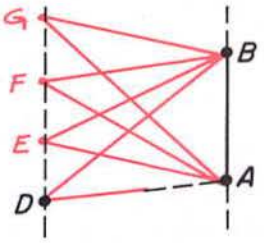
The number of spaces between r and s (or between C and D) is called the *Altitude* of the triangle, and the number of spaces between A and B is called the *Base* of the triangle.

Let a be the altitude of a triangle and b be the base. Then:

Area of a triangle $= \frac{1}{2}\, a \times b$ or $\frac{1}{2}\, ab$.

II.





On a line locate two points, A and B.

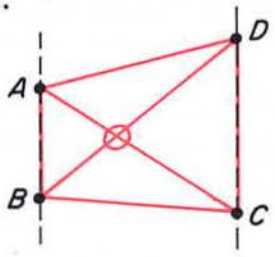
On a line parallel to line AB locate four points and label them D, E, F, and G.

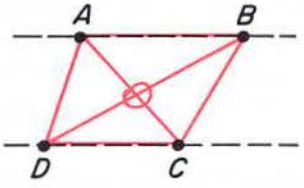
Draw triangles ABD, ABE, ABF, and ABG.

What can you say about the areas of the four triangles in any situation like that?

AREAS OF ALL FOUR TRIANGLES ARE EQUAL BECAUSE THEY HAVE THE SAME BASE AND SAME ALTITUDE.

III.





Locate points A and B on a line, and C and D on a line parallel to line AB, so that connecting A to D and B to C will create a quadrilateral with diagonals AC and BD crossing at O.

What can you say about the areas of $\triangle ABD$ and $\triangle ABC$? . . . and about the areas of $\triangle AOD$ and $\triangle BOC$?

AREA OF $\triangle ABD$ = AREA OF $\triangle ABC$. AREA OF $\triangle AOB$ IS COMMON TO BOTH. IF $\triangle AOB$ IS SUBTRACTED FROM BOTH, $\triangle AOD$ AND $\triangle BOC$ ARE LEFT — AND THEIR AREAS ARE EQUAL.

What can you say about the shapes of $\triangle AOB$ and $\triangle COD$?

THEY ARE SIMILAR. BECAUSE OF THE 'Z' RESULT, WE KNOW THAT EACH ANGLE IN ONE IS THE SAME SIZE AS AN ANGLE IN THE OTHER.

94 Exploring the formula Area of a triangle = 1/2 ab, where a is the altitude of the triangle and b is the base

Tasks: *Pupils study or draw three sets of geometric figures following directions given. They then (1) judge a mathematical statement for set I, (2) study the formulation for determining the area of a triangle, and (3) formulate statements for three* What Can You Say? *situations for sets II and III.*

Purpose: To help pupils observe significant relationships in sets of geometrical sketches. To extend the study of the idea of area.

Unifying Ideas: Functions and relations; geometry; measurement.

The Lesson: The sketches in I are described as if they were drawn on a piece of graph paper. However, only two of the lines, r and s, and only three of the points of intersection, A, B, and C, are indicated.

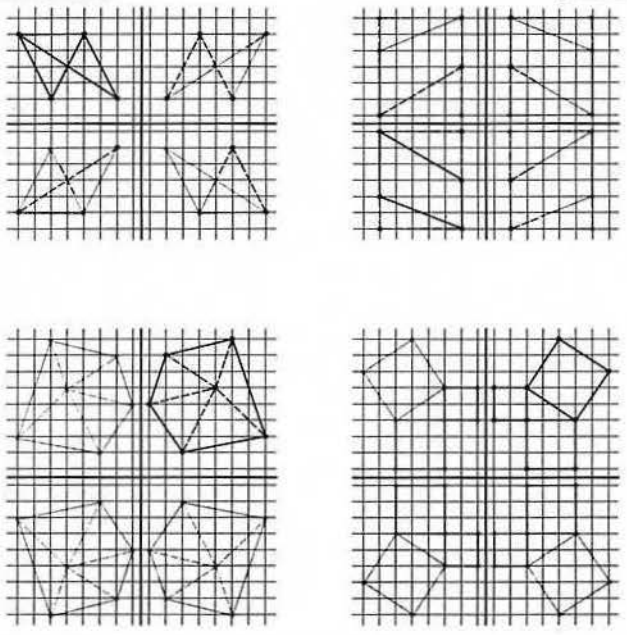
Would the line segment CD be part of one of the lines on the graph paper? Yes, because C is an intersection of two perpendicular lines and the line CD is drawn perpendicular to r and s. Would the point D also be a point of intersection on the graph paper? (Yes.)

Thus, line segments AB, AD, BD, and CD would each be a whole number of units in length. (There are certain triangles that could not be drawn according to these directions. One such is an equilateral triangle, a triangle all of whose sides are equal in length.)

We can, then, treat these triangles very much as we treated the triangles on pupil page 93.

The third set of sketches, of the quadrilateral with one pair of parallel sides, can be used to strengthen the notion that the area of a triangle is half the base times the altitude. The diagonals of a trapezoid (a quadrilateral with just two parallel sides) create four small triangles. The two opposite triangles that don't include either of the parallel sides have equal areas.

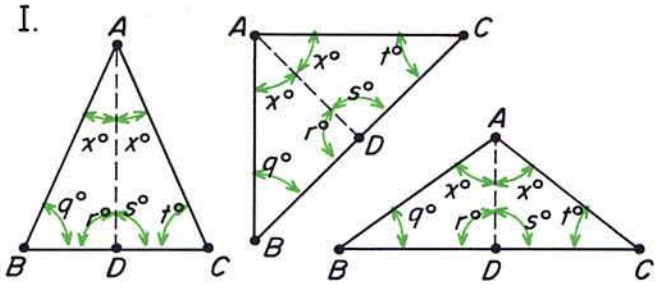
More Reflections EPS-94



Arrange 84 tiles in a rectangular array.
Always use as many of the 84 tiles as possible.

	a	b	c	d	e	f	g	h	i	j	k	l
No. of rows	8	7	15	5	21	9	4	6	17	7	29	3
No. of columns	10	12	5	16	4	9	19	13	4	11	2	23
No. of tiles used	80	84	75	80	84	81	76	78	68	77	58	69
No. of tiles not used	4	0	9	4	0	3	8	6	16	7	26	15

I.



These triangles are drawn so that, in each, sides AB and AC have the same length.

A line is drawn from A to the point labeled D on the opposite side; and line AD is drawn so that it bisects $\angle BAC$ — so $\angle BAD$ is the same size as $\angle CAD$.

What can you say that is true with regard to all three figures?

(1) Sides AB and AC have the same length. (T) F

(2) $\triangle ADB \cong \triangle ADC$ (T) F

(3) $r = s = 90$ (T) F

What else can you say that is true about all three figures?

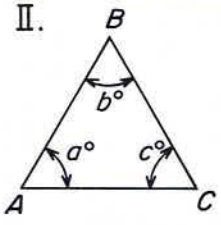
LINE SEGMENTS BD AND DC ARE THE SAME LENGTH. $q = t$.

LINE SEGMENT AD IS PERPENDICULAR TO LINE SEGMENT BC.

If a triangle has two sides of equal length, we call it an ISOSCELES triangle.

Because $q = t$ above, we say that in an isosceles triangle the angles opposite the sides of equal length are equal in size.

II.

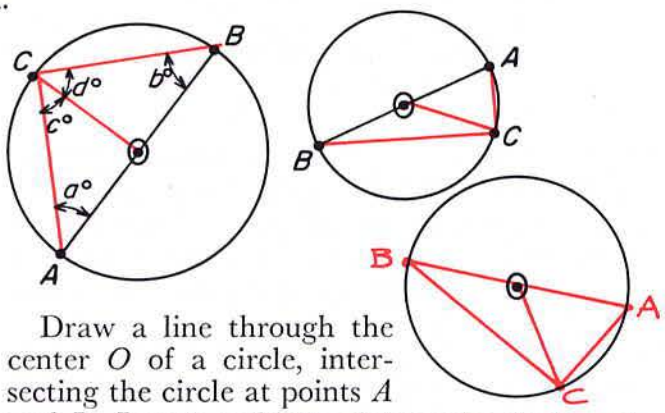


Sides AB, AC, and BC are all equal in length. If the three sides of a triangle are the same length, we call that an EQUILATERAL triangle.

What can you say about the angles of an equilateral triangle?

THEY ARE ALL EQUAL — $a = b = c = 60$ OR $\frac{180}{3}$.

III.



Draw a line through the center O of a circle, intersecting the circle at points A and B. Locate a third point on the circle and label it C. Draw triangle ABC. Draw line OC.

What can you say for all the sketches above?

(1) The distance between A and O is the same as the distance between O and B. (T) F

(2) Point O is just as far from A as it is from C. (T) F

(3) Point C is just as far from A as it is from B. T (F)

(4) $a = c$ (T) F

(5) $a = b$ T (F)

(6) $b = d$ (T) F

(7) $a + b = c + d$ (T) F

(8) $a + b + c + d = 180$ (T) F

(9) $a + b = 100$ T (F)

(10) $c + d = 90$ (T) F

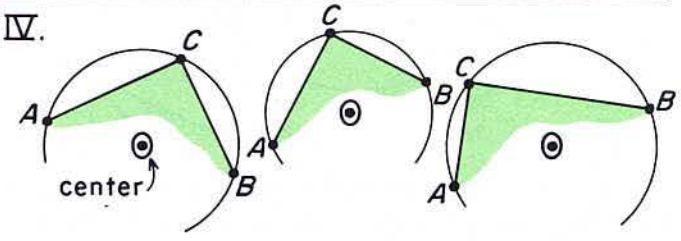
(11) $\triangle AOC \cong \triangle BOC$ T (F)

(12) The area of $\triangle AOC$ is the same as the area of $\triangle BOC$. T (F)

(13) Angle ACB is a right angle. (T) F

(14) $a + d = c + b$ (T) F

IV.



What can you say about line AB if $\angle ACB$ is a right angle?

IT PASSES THROUGH THE CENTER (IT'S A DIAMETER).

95 Exploring properties of isoceles triangles, equilateral triangles, and triangles inscribed in a semicircle

Tasks: *Pupils study or draw four sets of geometric figures. They then judge statements about these sketches or formulate true statements based on these sketches.*

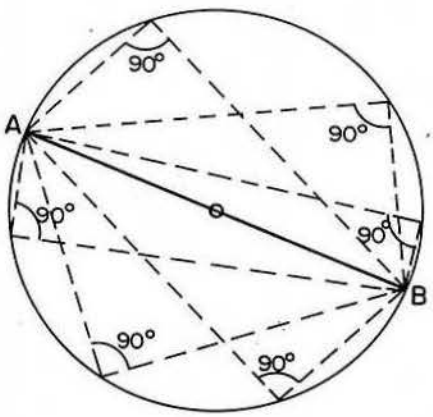
Purpose: To help pupils observe significant relationships in sets of geometric sketches. To extend the study of the geometric ideas of area and congruence, and to use properties of isosceles triangles.

Unifying Ideas: Functions and relations; geometry; measurement.

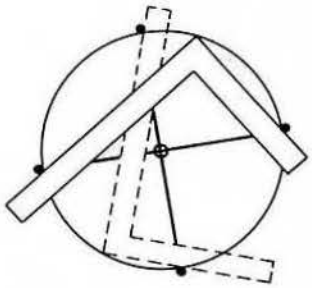
The Lesson: It is intuitively obvious that the angles opposite the congruent sides of an isosceles triangle are equal in size. Thus, the arguments on pupil page 95 are readily accepted.

In an isosceles triangle ABC (with side AB the same length as side AC) a line AD is drawn that bisects angle BAC. Comparing the triangles ABD and ACD, we know that sides AB and AC are the same length; AD is a side of both triangles; and the angles BAD and DAC are equal in size since AD was drawn as a bisector. So, triangles ABD and ACD are congruent; hence, angles ABD and ACD are the same size.

While this is quite apparent, we can use this result to find out something on pupil pages 95 and 96 that is not at all apparent. One side of a triangle is a diameter of a circle and the third vertex of the triangle is on the circle if and only if the triangle is a right triangle.

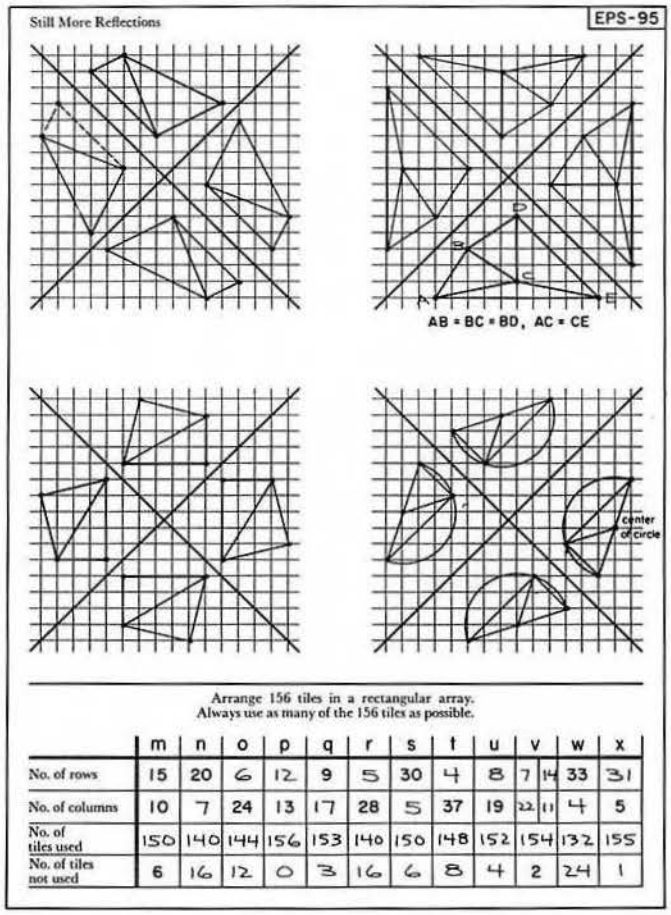


Carpenters sometimes use this idea to help find the center of a circle. They use a carpenter's square in much the same way as indicated at the bottom of the right-hand column to find the end points of two diameters:



They then connect these end points. When they mark the point where these two lines intersect they have marked the center of the circle.

Practice sheet 95 considers more reflections.



Arrange 156 tiles in a rectangular array.
Always use as many of the 156 tiles as possible.

	m	n	o	p	q	r	s	t	u	v	w	x
No. of rows	15	20	6	12	9	5	30	4	8	7	14	33	31
No. of columns	10	7	24	13	17	28	5	37	19	22	11	4	5
No. of tiles used	150	140	144	156	153	140	150	148	152	154	132	155
No. of tiles not used	6	16	12	0	3	16	6	8	4	2	24	1

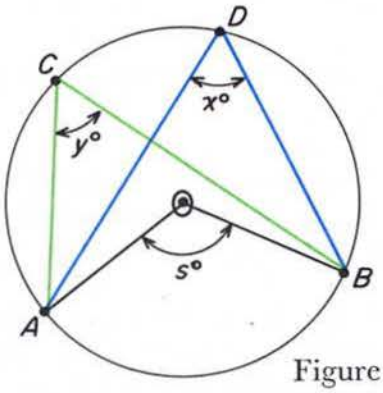


Figure 1

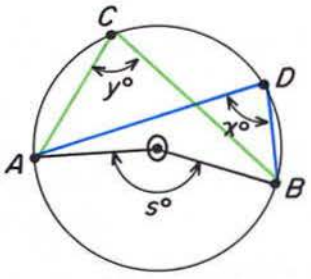


Figure 2

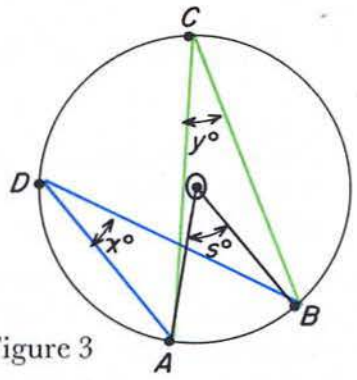


Figure 3

Points A and B are located on a circle, and a straight line is drawn from each to the circle's center, labeled O.

Two other points on the circle are labeled C and D. Lines from A and B are drawn to C and to D.

What can you say that's true about all the figures above?

Before making any statements, use a protractor to find y, x, and s in each figure and tabulate your results in the chart in the next column.

Number of Degrees

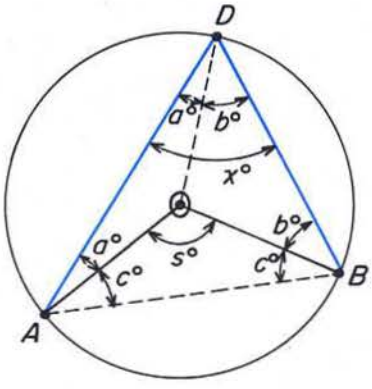
	x	y	s
Fig. 1	59	59	118
Fig. 2	80	80	160
Fig. 3	25	25	50

My conclusions are:

$S = 2x = 2y$ OR x AND y ARE EACH HALF OF S.

Here is another approach — a test of your conclusions based on experiments.

We shall consider $\angle AOB$ and $\angle ADB$ in Figure 1 above.



We draw lines connecting D and O, and connecting A and B.

Sides AO, BO, and DO are all the same length.

$\triangle DOA$, $\triangle AOB$, and $\triangle DOB$ are all isosceles triangles.

So, angles OAD and ODA are the same size; so are angles ODB and OBD and so are angles OBA and OAB.

$(c + a) + (a + b) + (b + c) = 180$

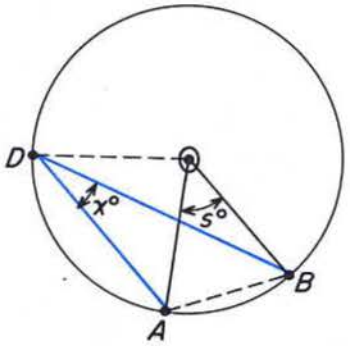
or ..... $2a + 2b + 2c = 180.$

But ..... $s + 2c = 180.$

So ..... $s = 2a + 2b = 2x.$

Does the argument on the left strengthen your belief in the conclusions you based on experiments?

However, do not jump to the conclusion that our argument is a complete proof. One of the examples above cannot be handled in the same way. Consider $\angle ABD$ and $\angle AOB$ in Figure 3.



Again, draw sides OD and BA.

Notice that sides OD, OA, and OB are all the same length.

Thus, angles OAD and ODA are the same size; so are angles ODB and OBD and so are angles OBA and OAB.

Try to proceed with an argument that will support the conclusion that $2x = s$.

96 Exploring the relationships between the degree measures of an inscribed angle and a central angle which subtend the same arc in a circle

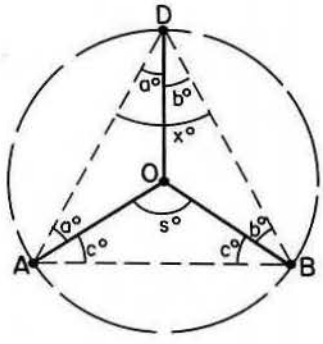
Tasks: *Pupils study a set of three figures, measure three angles in each figure, record the measures in a table, and formulate conclusions relating these angles. They then study an argument designed to support the conclusions resulting from measurement.*

Purpose: To help pupils observe significant relationships in a set of sketches. To extend the study of the applications of the fact that the angles opposite the congruent sides of an isosceles triangle are congruent.

Unifying Ideas: Functions and relations; geometry; measurement.

The Lesson: A less-than-obvious fact is considered on pupil page 96—and the argument given to support the idea again rests basically on the fact that the angles opposite the sides of equal length in an isosceles triangle are equal in size. Pupils are asked to find by measurement the numbers s, y, and x in each figure and record them in the chart. If their measurements are reasonably accurate, they will notice two rather surprising results: $x=y$ and $s=2x=2y$.

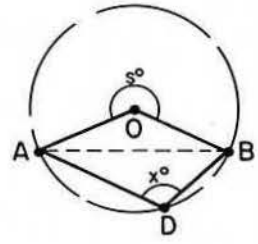
Before beginning a supporting argument for this relationship, we draw the lines from the center, O, to points A, B, and D. When we connect A, B, and D, we have three triangles; two sides in each of these triangles are radii of the circle and hence equal in length. Thus, we have three isosceles triangles.



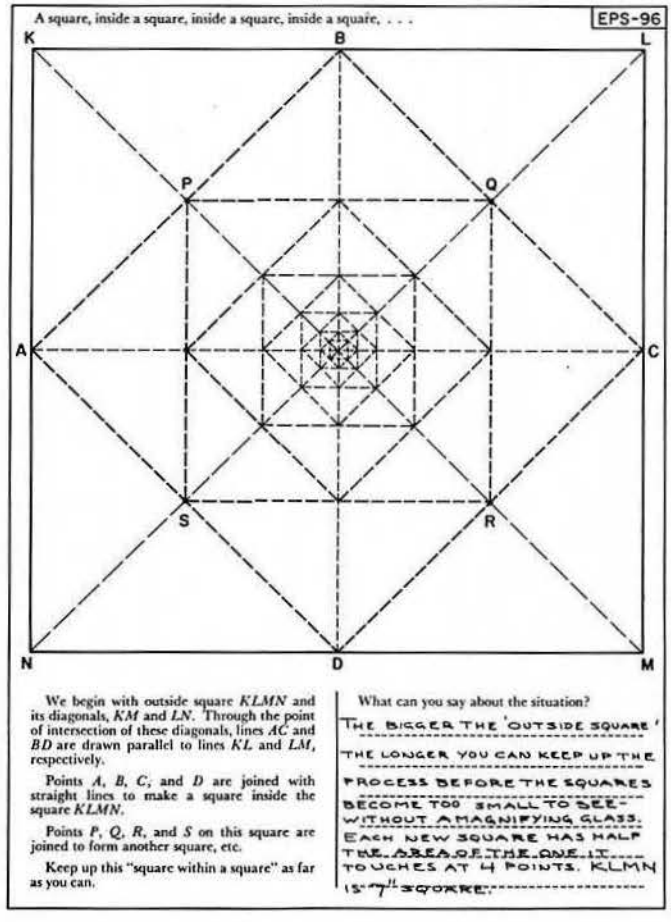
Now we have three pairs of congruent angles. We know that the sum of the angle-measures of $\triangle ADB$ is 180°, and we notice that $2a°+2b°+2c°$ is 180°. We know that, for $\triangle AOB$, $s°+2c°$ is 180°. So, $s°$ must be $2a°+2b°$. But, $2a°+2b°=2(a°+b°)=2x°$. Therefore, $s°=2x°$. This says precisely what the experiment suggested. The same argument would lead to the conclusion that $s°=2y°$. It follows, of course, that $x°=y°$.

An argument is a little harder to follow when the center of the circle does not fall inside angle ADB, but we have hinted at such an argument in the right-hand column. We do not expect all pupils to construct such an argument.

We have not considered the case in which the center of the circle, O, and point D are on different sides of the line AB:



In this case, the relationship still holds, but $s°$ is more than 180°. The argument in this case is quite similar, beginning with the drawing of line OD.



XIV Decimal Fractions

Primitive man may have invented units of length before inventing units for other quantities, using the length of his foot, the width of his hand, and, for finer measurement, the width of his thumb.

Man's need for fine measurement has kept pushing his ability to devise better instruments.

When automobile racing began in 1897, the record speed was reported to be 37.5 miles per hour. In 1965, when Craig Breedlove broke the record at Bonneville Flats, his average speed over a 2-mile course was 600.601 m.p.h. — 600 and 601 thousandths of a mile per hour.

Experience in the annual coast-to-coast Mobil Gas Economy Run has required reporting results in hundredths of miles per gallon.

Here are part of the results of one of the annual runs:

Intermediate Size V-8's	Miles per gallon
Dodge Coronet	20.76
Plymouth Belvedere	20.61
Buick Special	20.30
Oldsmobile F85	19.80
Comet 404	19.58
Ford Fairlane	19.28
Pontiac Tempest	19.11
Rambler Ambassador	18.44
Chevelle Malibu	18.17

What Can You Say? THE PERFORMANCE OF THE DODGE CORNET WAS 2.59 MILES PER GALLON BETTER THAN THAT OF THE CHEVELLE MALIBU.

Here are the cars that placed first, second, and third in their class:

Fullsize Medium Priced V-8's	miles per gallon
Pontiac Catalina	19.84
Pontiac Star Chief	19.64
Buick LeSabre	19.63

What Can You Say? THE PERFORMANCE OF THE PONTIAC CATALINA WAS .21 MILES PER GALLON BETTER THAN THAT OF THE BUICK LE SABRE.

In the 1965 professional baseball leagues, pitchers were ranked on the basis of the average number of earned runs they allowed in each 9 innings they pitched.

McDowell of Cleveland won top honors for the lowest earned run average (E.R.A.) of pitchers in the American League who pitched more than 200 innings — an E.R.A. of only 2.17 earned runs in each 9 innings.

Koufax of Los Angeles had the best record in the National League — an E.R.A. of only 2.04 for 9 innings.

What Can You Say? KOUFAX'S E.R.A. WAS .13 BETTER THAN Mc DOWELL'S.

On February 26, 1966, the Weather Bureau in Monterey, California, reported the amount of rainfall for the last 24 hours, the amount since July 1, 1965, and the total amount between July 1, 1964 and February 26, 1965. Here is part of the report. (The measurements are in inches.)

	Last 24 hrs.	This year	Last year
Carmel	.20	16.17	12.10
Marina	.30	14.82	12.13
Monterey	.23	16.69	14.22
Pacific Grove	.19	16.49	12.58
Salinas	.20	11.81	9.32
St. Clemente	.12	14.77	13.56

What Can You Say? THIS YEAR'S RAINFALL FOR ST. CLEMENTE IS 1.21 INCHES MORE THAN LAST YEARS OR APPROX. 10 TIMES THE AMOUNT OF RAINFALL IN THE LAST 24 HRS.

On June 3, 1960, Captain Yates flew a Boeing 707 from New York to San Francisco at the rate of 510.092 m.p.h. On December 8, 1961, Captain Miller flew a Boeing 707 from Los Angeles to New York at the rate of 636.8 m.p.h. Both were records.

What Can You Say? CAPT. MILLER SET HIS RECORD BY FLYING AT THE RATE OF 126.708 M.P.H. FASTER THAN CAPT. YATES.

97 Measurement reports in decimal fractions

Tasks: *Pupils read about applications of decimal fractions and formulate statements in response to* What Can You Say? *situations.*

Purpose: To help pupils understand that decimal fraction notation is a special language for expressing certain common fractions. To help pupils understand how this notation is used in everyday life.

Unifying Ideas: Numeration; measurement.

The Lesson: Decimal fractions are a particular shorthand that has been developed to write fractions that have powers of 10 as denominators—10, 100, 1000, etc. Pupils ought to understand that we are introducing nothing new except a language for expressing the fractions in a special group. Further, pupils already are acquainted at least with the notation as it is used in writing amounts of money and on the mileage indicators of automobiles.

Pupil page 97 suggests many other situations in which decimal fractions are used—in expressing speed records, in comparing gas consumption by cars, in comparing records of baseball pitchers, in charting amounts of rainfall, in commercial airline jet records.

Pupils may be able to suggest many other instances in which decimal fractions are used. Many supermarkets have machines that weigh packages of meats and cheese to the nearest hundredth of a pound and automatically stamp the weight and price on the package. Many clocks punch time cards in tenths and hundredths of an hour. Gasoline prices typically indicate tenths of a cent—33.9¢ per gallon. Dimensions on engineering drawings are almost always written as decimal fractions. In most libraries, books in a given subject area are assigned decimal fractions and arranged on the shelves accordingly. A meter stick is marked off so that lengths can easily be expressed in decimal fraction notation. The whole metric system, which is used almost exclusively in non-English speaking countries, has this same advantage.

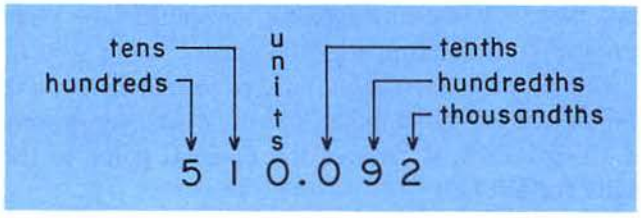
Without the shorthand we call "decimal fractions," we would have had to use expressions such as these:

$600\frac{601}{1000}$ $20\frac{76}{100}$ $636\frac{8}{10}$ etc.

instead of:

600.601 20.76 636.8

This shorthand uses the idea of place value to the right of units or ones place. Here's the plan:



Read it as five hundred ten and ninety-two thousandths.

As we learn more about this shorthand, we shall discover important shortcuts.

Fractions already written as tenths, hundredths, thousandths, etc., can be written directly in decimal fraction shorthand. Here are some examples:

$\frac{1}{10} = .1$ $\frac{8}{10} = .8$ $\frac{6}{10} = .6$

$\frac{17}{100} = .17$ $\frac{10}{100} = .10$ $\frac{57}{100} = .57$

$\frac{9}{100} = .09$ $\frac{1}{100} = .01$ $\frac{70}{100} = .70$

$\frac{9}{1000} = .009$ $\frac{11}{1000} = .011$ $\frac{125}{1000} = .125$

$\frac{5}{10} = .5$ $\frac{5}{100} = .05$ $\frac{5}{1000} = .005$

$\frac{9}{10} = .9$ $\frac{75}{100} = .75$ $\frac{36}{1000} = .036$

$\frac{8}{100} = .08$ $\frac{83}{1000} = .083$ $\frac{80}{1000} = .080$

$1\frac{6}{10} = 1.6$ $7\frac{3}{10} = 7.3$ $2\frac{18}{100} = 2.18$

$3\frac{8}{10} = 3.8$ $3\frac{8}{100} = 3.08$ $6\frac{235}{1000} = 6.235$

$125\frac{9}{10} = 125.9$ $7\frac{8}{100} = 7.08$ $7\frac{8}{1000} = 7.008$

$40\frac{7}{10} = 40.7$ $40\frac{7}{100} = 40.07$ $40\frac{70}{1000} = 40.070$

Some fractions can easily be rewritten as tenths, hundredths, thousandths, etc., and then written in decimal fraction shorthand. Here are some examples:

$\frac{1}{2} = \frac{5}{10} = .5$ $\frac{1}{4} = \frac{25}{100} = .25$

$\frac{1}{5} = \frac{2}{10} = .2$ $\frac{3}{4} = \frac{75}{100} = .75$

$\frac{3}{5} = \frac{6}{10} = .6$ $\frac{2}{5} = \frac{4}{10} = .4$

$\frac{1}{20} = \frac{5}{100} = .05$ $\frac{1}{25} = \frac{4}{100} = .04$

$\frac{1}{50} = \frac{2}{100} = .02$ $\frac{4}{5} = \frac{8}{10} = .8$

$\frac{3}{20} = \frac{15}{100} = .15$ $\frac{3}{25} = \frac{12}{100} = .12$

$\frac{3}{50} = \frac{6}{100} = .06$ $\frac{7}{20} = \frac{35}{100} = .35$

$\frac{1}{8} = \frac{125}{1000} = .125$ $\frac{3}{8} = \frac{375}{1000} = .375$

$\frac{7}{8} = \frac{875}{1000} = .875$ $\frac{5}{8} = \frac{625}{1000} = .625$

Some fractions can be reduced and then written as decimal fractions.

$\frac{2}{4} = \frac{1}{2} = .5$ $\frac{3}{12} = \frac{1}{4} = .25$

$\frac{12}{16} = \frac{3}{4} = .75$ $\frac{7}{14} = \frac{1}{2} = .5$

$\frac{3}{15} = \frac{1}{5} = .2$ $\frac{9}{36} = \frac{1}{4} = .25$

$\frac{2}{40} = \frac{1}{20} = .05$ $\frac{12}{30} = \frac{2}{5} = .4$

$\frac{3}{75} = \frac{1}{25} = .04$ $\frac{18}{24} = \frac{3}{4} = .75$

$\frac{28}{35} = \frac{4}{5} = .8$ $\frac{3}{24} = \frac{1}{8} = .125$

$\frac{18}{36} = \frac{1}{2} = .5$ $\frac{48}{50} = \frac{96}{100} = .96$

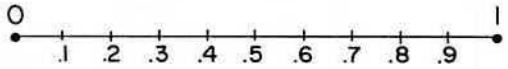
$\frac{33}{44} = \frac{3}{4} = .75$ $\frac{25}{40} = \frac{5}{8} = .625$

Tasks: *Pupils find decimal equivalents for fractions.*

Purpose: To help pupils understand the relative size of tenths, hundredths, and thousandths. To help pupils understand how fractions can be expressed in decimal fraction notation.

Unifying Ideas: Numeration; functions and relations.

The Lesson: To help pupils visualize the relationship of units, tenths, hundredths, and thousandths, you might draw a line from one end of the blackboard to the other. Mark two points 100 inches apart, labeling them with a 0 and a 1. Let that distance be considered the unit. Mark off every 10 inches and label these points:



Marks 1 inch apart will divide the unit into hundredths, but there will not be enough room to label them.

Finally, consider marking off thousandths! Even though the unit chosen is more than 8 feet long, marks for thousandths would be less than $\frac{1}{8}$ of an inch apart and would present nothing but a blur to those in the back of the room. (Even if the class decides on a unit twice as long—almost 17 feet—the marks for thousandths will still be almost illegibly close together.)

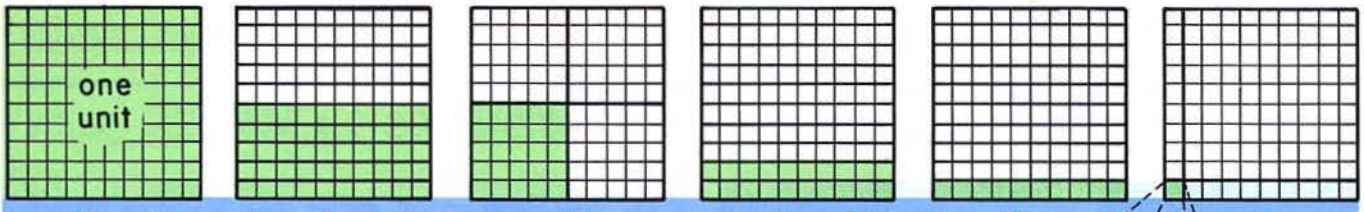
Such an activity helps develop a feeling for the relative size of tenths, hundredths, and thousandths.

Now, as various exercises in finding decimal equivalents are considered, points can be marked on this giant-size number line, with both the fraction and the decimal labels.

With only the marks for tenths labeled, consider placing marks for such numbers as $\frac{9}{100}$, $\frac{17}{100}$, $\frac{57}{100}$, $\frac{75}{100}$, and $\frac{83}{100}$. "Between what two labels shall we make a mark for $\frac{9}{100}$? $\frac{17}{100}$? $\frac{57}{100}$?" (Between the labels for 0 and .1; between the labels for .1 and .2; between the labels for .5 and .6, etc.)

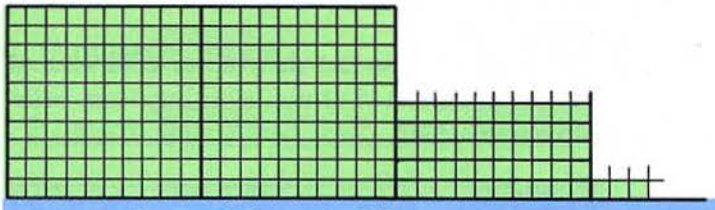
"Is $\frac{9}{100}$ closer to 0 or to .1? Is $\frac{75}{100}$ closer to .7 or to .8?"

Next, consider the numbers $\frac{9}{1000}$, $\frac{11}{1000}$, $\frac{78}{1000}$, and $\frac{125}{1000}$. "Where should we make marks for these numbers?" Since only the tenths and the hundredths are marked, all of the given numbers will be located *between* marks. In describing these locations by saying things such as "$\frac{11}{1000}$ would be just a little bit to the right of the first mark after 0" and "$\frac{125}{1000}$ would be halfway between .12 and .13," pupils will develop a deeper understanding of this notation—$\frac{11}{1000}$ is a little larger than .01; .12 is a little more than .1; .125 is more than .12 but less than .13; etc.



$1 = \frac{10}{10} = \frac{100}{100}$ $\frac{1}{2} = \frac{5}{10} = \frac{50}{100}$ $\frac{1}{4} = \frac{25}{100}$ $\frac{1}{5} = \frac{2}{10} = \frac{20}{100}$ $\frac{1}{10} = \frac{10}{100} = .1 = .10$ $\frac{1}{100}$

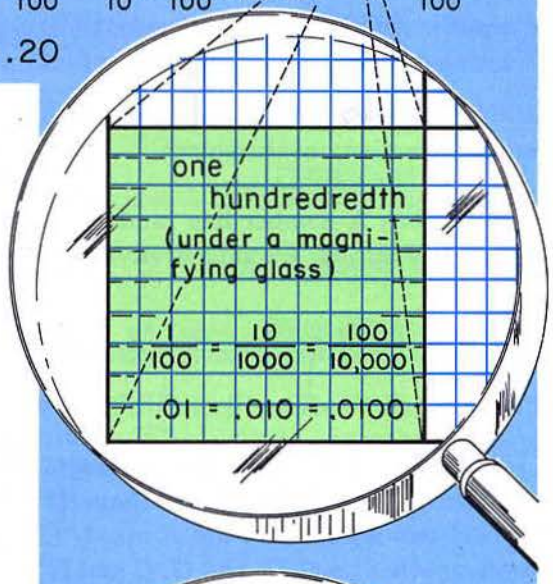
1 = 1.0 = 1.00 .5 = .50 .2 = .20

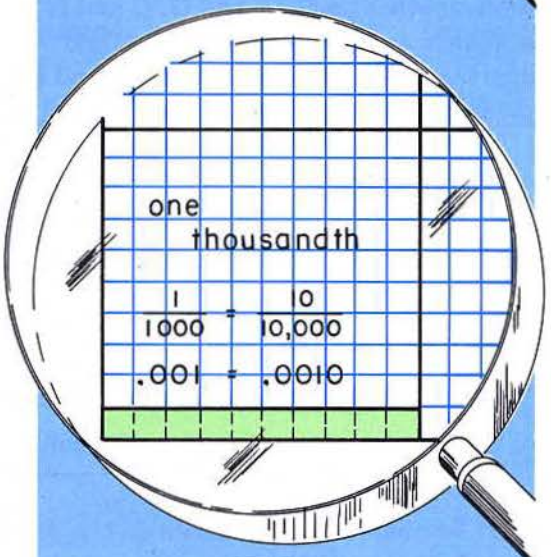


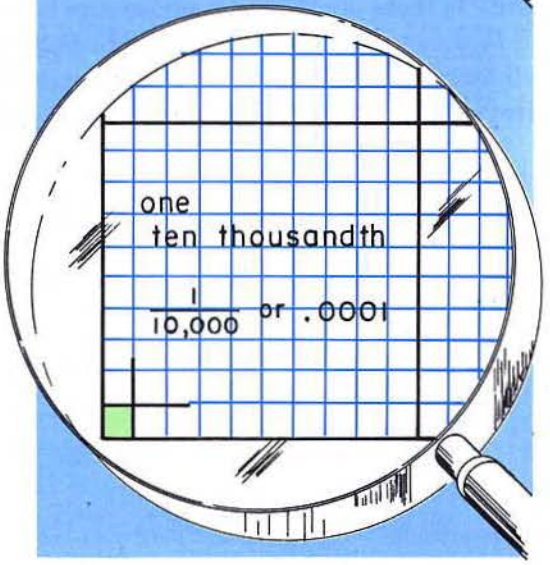
2 + .5 + .03 = 2.53

Two <u>and</u> fifty-three hundredths

As we read decimal shorthand the word "and" signals the decimal point at the right of the units place.







What Are My Rules?

	3	15	1
x 100	300	1,500	100
x 10	30	150	10
x 1	3	15	1
÷10	.3	1.5	.1
÷100	.03	.15	.01

	.4	8.7	.25
x 100	40	870	25
x 10	4	87	2.5
x 1	.4	8.7	.25
÷10	.04	.87	.025
÷100	.004	.087	.0025

1.7	x 3	5.1	÷100	.051
25	x 15	375	÷100	3.75
.18	x 4	.72	÷10	.072

99 "What Are My Rules?"

Tasks: *Pupils study diagrams which illustrate the relationships among units, tenths, hundredths, thousandths, and ten thousandths, and complete charts involving multiplication and division of whole numbers and decimals.*

Purpose: To help pupils understand the relative size of decimals through pictures and numerical patterns.

Unifying Ideas: Number and counting; structure; numeration; functions and relations; multiplication and division; geometry; measurement.

The Lesson: We try on pupil page 99 to suggest the relationships among units, tenths, hundredths, thousandths, and ten thousandths by employing a magnifying glass. However, we suggest you consider drawing the largest square you can on the chalkboard. If you can make a square 1 meter on each side then the hundredths block in your drawing will show "under the magnifying glass." Such a sketch would help considerably to demonstrate the relationship we are discussing.

Or perhaps someone or a group could make a large drawing of a similar nature, including lines to divide the large unit into ten-thousandths. This would probably be the first time many of the pupils would have seen 10,000 of anything, or would have seen something divided into 10,000 parts.

The simple exercises on pupil page 99 demonstrate the results of multiplying (or dividing) by powers of 10—in effect, it "moves the decimal point to the right (or left)."

Chain Reactions

I.

(a) $1.50 +7 dimes $2.20 −37¢ $1.83 +half a dollar $2.33 +$.80 $3.13

(b) 7.5 miles +.6 mi. 8.1 mi. +.9 mi. 9 mi. −.7 mi. 8.3 mi. +.25 mi. 8.55 mi.

(c) 5.25 in. +.5 in. 5.75 in. −3 in. 2.75 in. +.6 in. 3.35 in. −0.5 in. 2.85 in.

(d) $.70 +$.45 $1.15 −25¢ $.90 +$7 $7.90 +1 dime $8.00

(e) .75 miles −.05 mi. .7 mi. +.3 mi. 1.00 mi. −.37 mi. .63 mi. +.9 mi. 1.53 mi.

(f) 2 hours +1.5 hrs. 3.5 hrs. −1.7 hrs. 1.8 hrs. −.25 hrs. 1.55 hrs. +8 hrs. 9.55 hrs.

(g) $3.00 −$1.83 $1.17 −50¢ $.67 +$.90 $1.57 +$1.57 $3.14

II. The new family car had gone only $17\frac{1}{2}$ miles before the trip began. Figure out the distance between each town and the next.

Tens of thousands of miles	Thousands of miles	Hundreds of miles	Tens of miles	Unit miles	Tenths of a mile		Distance in miles between towns
0	0	0	1	7	5	Home	10.0
0	0	0	2	7	5	Malta	20.4
0	0	0	4	7	9	Troy	2.1
0	0	0	5	0	0	Dover	35.2
0	0	0	8	5	2	Union	14.7
0	0	0	9	9	9	Lee City	8.1
0	0	1	0	8	0	Bureau	67.3
0	0	1	7	5	3	Olan	24.7
0	0	2	0	0	0	Benton	67.8
0	0	2	6	7	8	Rapids	9.9
0	0	2	7	7	7	Knox	23.6
0	0	3	0	1	3	Hull	

The shortest distance between two towns was 2.1 miles. The longest distance between neighboring towns was 67.8 miles. The total length of the trip was 283.80 miles.

Distance between			(in miles)
Malta	and	Dover	22.5
Dover	and	Lee City	49.9
Lee City	and	Benton	100.1
DOVER	and	BUREAU	58.0
BENTON	and	HULL	101.3
HOME	and	DOVER	32.5
DOVER	and	OLAN	125.3
HOME	and	OLAN	157.8
MALTA	and	UNION	57.7
MALTA	and	KNOX	250.2
UNION	and	KNOX	192.5

A round trip between Olan and Benton is 49.4 miles. A round trip between UNION and LEE CITY is 29.4 miles. Half of the distance between BENTON and RAPIDS is 33.9 miles.

In 698.7 more miles, the speedometer will read 1000.0.

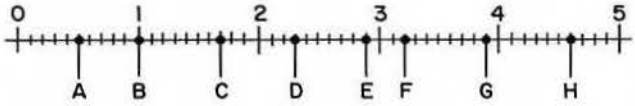
100 Chain Reactions (Addition and subtraction with decimal notation)

Tasks: *Pupils complete chain reaction problems and compute distances between towns.*

Purpose: To provide practice in adding and subtracting decimals in measurement situations.

Unifying Ideas: Structure; numeration; addition and subtraction; measurement.

The Lesson: A number line randomly labeled with letters will initiate a good activity.



Find two letters that are 1.2 apart. (*A* and *C*) Is there another pair that are 1.2 apart? (No)

Find two letters that are .7 apart. (*B* and *C*) Is there another pair? (*F* and *G*, *G* and *H*)

Which are farthest apart? (*A* and *H*) How far apart? (4.1) Which are closest? (*E* and *F*) Are there three points equally spaced? (*F*, *G*, and *H*)

There are many questions like these that require considerable searching and computing.

Ask pupils to mention amounts of money less than $2, and list them on the board.

73¢	a quarter	three dimes	$1.24	
7 nickels	3 half-dollars	$.69	$.48	
39 cents	three-quarters of a dollar			

Find two amounts whose sum is 78¢. (3 dimes and 48¢) Is there another pair whose sum is 78¢?

It is soon obvious that it would be helpful to write all these amounts in the same notation, and in order, from smallest to largest perhaps.

$.25	$.30	$.35	$.39	$.48	$.69
$.73	$.75	$1.24	$1.50		

Find a pair whose sum is $.85. (There isn't any.) Is there a pair whose difference is $.85? (Yes, $1.24 and $.39)

Is every item in the list the sum or difference of other items in the list—but not using any item twice in one combination? .

$.25 = $.73 − $.48
$.30 = $.69 − $.39
$.35 = $.78 + $.30 − $.48 − $.25
$.39 = $.69 − $.30
etc.

There are many, many variations of this activity—based on lists provided either by the pupils or the teacher. Each pupil ought to have an opportunity to work by himself in finding combinations. Otherwise, quick searchers will discourage those who work more slowly and who might work more methodically if given enough time.

The lower section of pupil page 100 is a variation of the idea above. Pupils compute distances between neighboring towns and later find two towns that are, say, 101.3 miles apart. One approach is to find two odometer readings 101.3 apart; another is to find consecutive distances between towns whose sum is 101.3:

301.3 − 200.0 = 101.3

or

67.8 + 9.9 + 23.6 = 101.3

I. A Different Kind of Chain Reaction.

(a) $2.50 | x 3 | $7.50 | ÷ 10 | $.75

(b) 8 in. | x 9 | 72 in. | ÷100 | .72 in.

(c) 1.5 hrs. | x 12 | 18 hrs. | ÷ 10 | 1.8 hrs.

(d) 1.2 miles | x 12 | 14.4 mi. | ÷100 | .144 mi.

(e) $ 415 | x 5 | $2,075 | ÷100 | $20.75

(f) .3 in. | x 3 | .9 in. | ÷ 10 | .09 in.

(g) .7 min. | x 13 | 9.1 min. | ÷100 | .091 min.

(h) $24.25 | x 6 | $145.50 | ÷100 | $1.455

In Alec's style:

(i)	8 lbs.	x 7 ÷ 10	5.6 lbs.	(j)	12 ft.	x 5 ÷ 10	6 ft.	(k)	1.8 mi.	x 8 ÷ 10	1.44 mi.
(l)	$ .05	x 9 ÷ 10	$.045	(m)	16 hrs.	x 12 ÷ 10	19.2 hrs.	(n)	16 hrs.	x 12 ÷100	1.92 hrs.
(o)	13.5 in.	x 15 ÷100	2.025 in.	(p)	$ 70.28	x 3 ÷100	$2.1084	(q)	1.6 min.	x 16 ÷ 10	2.56 min.
(r)	$ 125	x 9 ÷100	$11.25	(s)	.3 mi.	x 3 ÷100	.009 mi.	(t)	$ 765	x 1 ÷1000	$.765

II. Multiplying fractions:

(a) $7 \times \frac{1}{10} = \frac{7}{10}$ (b) $\frac{3}{10} \times 3 = \frac{9}{10}$ (c) $\frac{7}{10} \times \frac{9}{10} = \frac{63}{100}$ (d) $\frac{13}{10} \times \frac{3}{100} = \frac{39}{1000}$

(e) $\frac{11}{10} \times \frac{11}{10} = \frac{121}{100}$ (f) $\frac{3}{10} \times \frac{7}{10} = \frac{21}{100}$ (g) $\frac{1}{10} \times \frac{19}{100} = \frac{19}{1000}$ (h) $\frac{31}{10} \times \frac{4}{100} = \frac{124}{1000}$

The same examples written in decimal shorthand:

(a) 7 × .1 = .7 (b) .3 × 3 = .9 (c) .7 × .9 = .63 (d) 1.3 × .03 = .039

(e) 1.1 × 1.1 = 1.21 (f) .3 × .7 = .21 (g) .1 × .19 = .019 (h) 3.1 × .04 = .124

(i) $\frac{25}{10} \times \frac{18}{10} = \frac{450}{100}$

2.5 × 1.8 = 200 + 25 = 4.50

(j) $\frac{173}{100} \times \frac{31}{10} = \frac{5363}{1000}$

1.73 × 3.1 = 173 + 519 = 5.363

(k) $3\frac{7}{10} \times \frac{5}{10} = \frac{185}{100}$

3.7 × .5 = 1.85

(l) $197 \times \frac{9}{100} = \frac{1773}{100}$

197 × .09 = 17.73

(m) .5 × 3 = 1.5

(n) 2.5 × .6 = 1.50

(o) 12.5 × .12 = 250 + 125 = 1.500

(p) 6.25 × .24 = 2500 + 1250 = 1.5000

101 A different kind of chain reaction (Multiplication with decimal notation)

Tasks: *Pupils complete problems involving multiplication of fractions.*

Purpose: To provide practice in multiplication of fractions using "common" notation and decimal notation.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations; measurement.

The Lesson: The "pairs of operations" and "chain reactions" should be recalled from the development dealing with fractions in earlier pages. In this case, however, the division operations are restricted to division by a power of 10—10, 100, or 1000:

$\left|\begin{matrix}\times 7 \\ \div 10\end{matrix}\right|$ and $\times \frac{7}{10}$ and $\times .7$

These are simply different notations for the same idea. Whenever pupils are in doubt, they should be encouraged to use the notation that gives them the greatest feeling of confidence.

After pupils have completed examples *a-h* in II, ask them to list the products from smallest to largest:

$\frac{19}{1000}$ $\frac{39}{1000}$ $\frac{124}{1000}$ $\frac{21}{100}$ $\frac{63}{100}$ $\frac{7}{10}$ $\frac{9}{10}$ $\frac{121}{100}$

and to do the same with the decimal equivalents:

.019 .039 .124 .21 .63 .7 .9 1.21

If pupils have difficulty with such ordering problems, you can be sure that their grasp of the fundamental notions is insufficient to help them through addition, subtraction, multiplication, and division involving decimals. So hesitancy at this point is a sure sign to go slowly, utilizing every opportunity for more experience in labeling number lines and in drawing sketches.

Practice in ordering groups of decimal fractions is not lost on those who have a firmer grasp on the basic ideas. These pupils will invent all kinds of little tricks to simplify the ordering. Each of these devices helps develop a facility that leads to complete confidence.

After the last eight examples (*i-p* of II) are completed, consider any shortcuts that are reliable guides to checking to see that the decimal point is properly placed. The following mnemonic devices may help:

units × tenths = tenths
tenths × units = tenths
tenths × tenths = hundredths
etc.

Someone will probably suggest noting the number of digits following the decimal point ("decimal places") in each of the two factors, adding these numbers of digits, and counting off that many decimal places in the result starting at the rightmost digit. Why is this reliable?

Another depends on estimating. In the example of 2.5×1.8, the result will be in the neighborhood of 4 or 5—certainly not .450 or 45.0 or 450. In the example 1.73×3.1, the result will be in the neighborhood of 5 or 6—certainly not .5363 or 53.63 or 536.3.

Most people use a combination of these two notions—a simple rule and common sense.

I. Undoing Multiplication . . . or, Dividing

1. $5 \times 3 = 15$ $3 \overline{)15}$ → 5	2. $7 \times .4 = 2.8$ $.4 \overline{)2.8}$ → 7	3. $.27 \times 5 = 1.35$ $5 \overline{)1.35}$ → .27	4. $1.8 \times .7 = 1.26$ $.7 \overline{)1.26}$ → 1.8	5. $1.4 \times .08 = .112$ $.08 \overline{).112}$ → 1.4	6. $.09 \times .6 = .054$ $.6 \overline{).054}$ → .09
7. $2.16 \times 3 = 6.48$ $3 \overline{)6.48}$ → 2.16	8. $1.74 \times .5 = .870$ $.5 \overline{).870}$ → 1.74	9. $29.6 \times .02 = .592$ $.02 \overline{).592}$ → 29.6	10. $85 \times .6 = 51.0$ $.6 \overline{)51.0}$ → 85.	11. $.55 \times .4 = .220$ $.4 \overline{).220}$ → .55	12. $7.5 \times .8 = 6.00$ $.8 \overline{)6.00}$ → 7.5
13. $.25 \times 4 = 1.00$ $4 \overline{)1.00}$ → .25	14. $.5 \times 2 = 1.0$ $2 \overline{)1.0}$ → .5	15. $.125 \times 8 = 1.000$ $8 \overline{)1.000}$ → .125	16. $.75 \times 4 = 3.00$ $4 \overline{)3.00}$ → .75	17. $.375 \times 8 = 3.000$ $8 \overline{)3.000}$ → .375	18. $.875 \times 8 = 7.000$ $8 \overline{)7.000}$ → .875

19. $2.5 \times 1.7 = 4.25$ $1.7 \overline{)4.25}$ → 2.5 175 25 34 85 85	20. $370 \times .43 = 159.10$ $.43 \overline{)159.10}$ → 370. 1110 1480 129 301 301	21. $450 \times .18 = 81.00$ $.18 \overline{)81.00}$ → 450. 3600 450 72 90 90	22. $62 \times .50 = 31.00$ $.50 \overline{)31.00}$ → 62. 300 100 100	23. $500 \times .014 = 7.000$ $.014 \overline{)7.000}$ → 500. 2000 500 70

II.

	Purchase price shown on the cash register	Money given to the clerk	Amount of change
a.	$00075	$1.00	$.25
b.	$00110	$5.00	$3.90
c.	$01250	$20.00	$7.50
d.	$00507	$10.00	$4.93
e.	$00117	$5.00	$3.83
f.	$05818	$60.00	$1.82
g.	$02746	$100.00	$72.54
h.	$03081	$37.00	$6.19
i.	$00879	$10.00	$1.21
j.	$02191	$25.00	$3.09
k.	$48629	$1,000.00	$513.71

102 Undoing multiplication (Division with decimal notation)

Tasks: *Pupils complete multiplication and division examples in the upper part of the page and provide missing information in examples at the bottom of the page.*

Purpose: To help pupils understand the relationships between multiplication and division and between addition and subtraction.

Unifying Ideas: Structure; multiplication and division; addition and subtraction; measurement.

The Lesson: Each example is given as a multiplication and as a division. Urge pupils to use every common-sense measure they know to reach reasonable results.

You may wish to tell the class about Barney's system, which he called the *multiply 'em both by whatever you need to rule.* "Both" refers to the dividend and the divisor, and that the "whatever you need to" refers to a multiplication that would produce a whole-number divisor.

He might tackle 2.8 ÷ .4 in several ways:

$4 \times 5 \overline{)2.8 \times 5} \rightarrow 2 \overline{)\overset{7}{14}}$

$4 \times 10 \overline{)2.8 \times 10} \rightarrow 4 \overline{)\overset{7}{28}}$

$4 \times 2.5 \overline{)2.8 \times 2.5} \rightarrow 1 \overline{)\overset{7}{7}}$

$4 \times 100 \overline{)2.8 \times 100} \rightarrow 40 \overline{)\overset{7}{280}}$

Is this a reliable method? Try it on another example, such as .112 ÷ .08:

$.08 \times 50 \overline{).112 \times 50} \rightarrow 4 \overline{)\overset{1.4}{5.6}}$

$.08 \times 100 \overline{).112 \times 100} \rightarrow 8 \overline{)\overset{1.4}{11.2}}$

$.08 \times 125 \overline{).112 \times 125} \rightarrow 10 \overline{)\overset{1.4}{14.0}}$

Finally, he refined his system, multiplying only by 10, 100, 1000, or some other power of 10. Why? Simply to have the easiest multiplying to do:

$.7 \overline{)1.26} \rightarrow .7 \times 10 \overline{)1.26 \times 10} \rightarrow 7 \overline{)12.6}$

$.02 \overline{).592} \rightarrow .02 \times 100 \overline{).592 \times 100} \rightarrow 2 \overline{)59.2}$

$.014 \overline{)7} \rightarrow .014 \times 1000 \overline{)7 \times 1000} \rightarrow 14 \overline{)7000}$

Barney justifies his method this way. You can indicate any division problem in common fraction notation. So the division problem:

$.02 \overline{).592}$

can be indicated by the fraction:

$$\frac{.592}{.02}$$

This fraction can be transformed as follows:

$$\frac{.592}{.02} = \frac{.592}{.02} \times 1 = \frac{.592}{.02} \times \frac{100}{100} = \frac{.592 \times 100}{.02 \times 100} = \frac{59.2}{2}$$

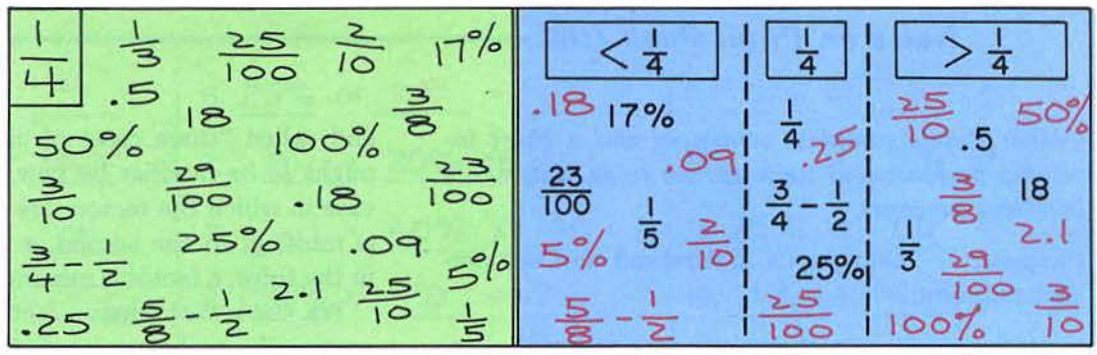
So, the original division problem is equivalent to the new division problem:

$2 \overline{)59.2}$

Ask the class if they can see either an improvement on Barney's system or another system that might be better. If the pupils don't come up with any better idea, we urge teachers to explain their own system and recommend it for consideration as reliable and useful.

I. Another setting on Alec's machine gave these results.

Find the pattern and complete the blue section.



II. To complete the record, you need to know what the shorthand "%" means. The percent sign is just another sign for "hundredths." Maybe it had a history like this:

$\frac{25}{100} \rightarrow \frac{25}{o/o} \rightarrow \frac{25}{\%} \rightarrow 25\%$

To say "25%" amounts to saying "25/100." So, we can say:

$\frac{25}{100} = \frac{1}{4} = .25 = 25\% = \frac{2}{8} = \frac{5}{20}$ etc.

$\frac{50}{100} = \frac{1}{2} = .50 = .5 = 50\% = \frac{15}{30}$

$\frac{1}{5} = \frac{2}{10} = \frac{20}{100} = 20\% = .20 = \frac{3}{15}$

$\frac{3}{4} = \frac{30}{40} = \frac{75}{100} = 75\% = .75 = \frac{12}{16}$

III. Complete the following, using:

= is equal to
< is less than
> is greater than

$\frac{1}{2} = 50\%$ $\frac{1}{4} < 30\%$ $10\% = \frac{10}{100}$

$\frac{1}{3} > 25\%$ $\frac{2}{5} = 40\%$ $75\% < .8$

$90\% = \frac{90}{100}$ $100\% = \frac{100}{100}$ $100\% = 1$

$125\% = 1\frac{1}{4}$ $150\% < 1\frac{3}{4}$ $200\% > 1\frac{1}{2}$

$23\% < \frac{1}{4}$ $\frac{3}{4} > 72\%$ $1\% < .1$

$15\% = \frac{3}{20}$ $\frac{1}{4} < 30\%$ $1\% < \frac{1}{2}$

IV. Here are some common uses of this shorthand:

25% off on all items.
We pay 3% on all savings deposits.
We won 50% of our games.
35% Dacron, 65% cotton.
100% Beef Frankfurters.
Pay only 75% of marked price.

What do these sentences tell you?

V. Below, write sentences that include the following expressions:

50% 10% 4% 125%

I HAVE READ 50% OF THE BOOK. '10% OFF' IS THE SAME AS 'ONE-TENTH OFF.' THE CLOTH MAY SHRINK 4%. IT USUALLY TAKES ME AN HOUR TO CUT THE LAWN, BUT TODAY IT WAS WET, AND IT TOOK ME 125% OF MY USUAL TIME. 50% IS 5 TIMES AS MUCH 10%. 4% OF $2.50 IS A DIME.

103 Percent notation and its meaning

Tasks: *Pupils follow the pattern in I, complete the sentences in II and III, discuss the meaning of common expressions using percent notation, and write sentences using percent notation.*

Purpose: To help pupils understand that percent notation is a shorthand for fractions. To help pupils understand the uses of percent notation.

Unifying Ideas: Numeration; functions and relations; measurement.

The Lesson: Percent—long a mysterious land of "Three Cases," burdened by intricate rules and regulations—is, in fact, nothing more than a shorthand for "common" fractions. The simple fact is that "¼," ".25," and "25%" are three notations for the same idea.

To make any more of the matter is to lead pupils into confusion. Like common fraction notation, percent notation can be used in many different ways, depending on the context. Presented with an expression such as "25% of three dollars," we shall think of 25 out of each 100 cents; and, since there are 300 cents, we think of $25¢ \times 3$, or 75¢. However, if someone said that 25% of a class of 24 pupils was absent, we would not think of the class as divided into 100 parts, 25 of which were absent; rather, we would think of ¼ of 24 pupils, concluding that 6 were absent.

Exercise II considers many ways to express such fractions as ¼, ½, ⅕, and ¾. We encourage you to extend this activity as far as you feel it necessary, such as:

$$\frac{1}{10} = 1 = .10 = 10\% = \frac{2}{20} = \frac{7}{70} \text{ etc.}$$

$$\frac{35}{100} = .35 = 35\% = \frac{7}{20} = .350 = \frac{21}{60} \text{ etc.}$$

$$\frac{9}{10} = \frac{90}{100} = .9 = .90 = 90\% = \frac{45}{50} \text{ etc.}$$

$$1\frac{1}{4} = \frac{5}{4} = \frac{125}{100} = 1.25 = 125\% = \frac{10}{8} \text{ etc.}$$

Again, asking pupils to order numbers from smallest to largest will help to indicate their grasp on the basic idea. Arrange the following from smallest to largest:

$$25\%, \frac{2}{5}, \frac{1}{10}, 8\%, 80\%, 2.5, 150\%$$

A variation of this activity is given in exercise III.

Discuss other ways of saying "25% off on all items" and the other sentences given in exercise IV. (Pay only ¾ of the price; prices cut by ¼; pay only 75% of the regular price; etc.)

Pupils may be asked to collect examples of everyday uses of percent notation from signs they see, from newspaper ads, or from magazines and books they read. Then have them discuss alternate ways of expressing the same ideas.

Keep the discussion focused as fully as possible on situations involving everyday uses of percent notation and on the fact that these situations could just as well be described using common fraction notation or decimal notation.

I. Complete the following sentences.

1. 50% of 38 = 19	6. 25% of 28 = 7	11. 75% of 4 = 3
2. 8 = 10% of 80	7. 2 = 20% of 10	12. 1 = 5% of 20
3. 25% x 36 = 9	8. 16 = 40% x 40	13. 75% x 80 = 60
4. 15 = 15% x 100	9. 150% x 20 = 30	14. 12 = 150% x 8
5. 1.5 = 15% of 10	10. 50 = 200% x 25	15. 20 % x 20 = 4

II. Each box below is to contain a "part" of the amount on the left. The relationship of that part to the whole amount is indicated at the top of the column.

	50%	25%	10%	75%	5%	150%
a. 1 hour	30 min.	15 min.	6 min.	45 min.	3 min.	90 min.
b. 1 yard and 4 inches	20 in.	10 in.	4 in.	30 in.	2 in.	60 in.
c. 5 dollars	$2.50	$1.25	$.50	$3.75	$.25	$7.50

	10%	20 %	15%	110%	130%	250 %
d. $25.00	$2.50	$5.00	$3.75	$27.50	$32.50	$62.50
e. 2 minutes	12 sec.	24 sec.	18 sec.	132 sec.	156 sec.	5 min.
f. 10 feet	1 ft.	2 ft.	18 in.	11 ft.	13 ft.	25 ft.

III. Write an appropriate story for each of the three headlines given below.

a. 40 % x 25 = 10

MARGE MISSED 40% OF THE 25 PROBLEMS. SHE HAD 10 WRONG AND 15 RIGHT. HENRY SAVED 40% OF HIS 25¢-A-WEEK ALLOWANCE. HE SAVED 10¢ A WEEK.

b. 10% x 70 = 7

BECAUSE OF A '10% OFF' SALE, I SAVED 7¢ ON THE BOOK. THE REGULAR PRICE OF THE BOOK WAS $.70. MY GOAL IS TO SAVE $70. I'VE SAVED $7, OR 10% OF THIS AMOUNT.

c. 25 % of 9 = 2.25

AFTER BILL JONES HAD GONE $2\frac{1}{4}$ MILES ON HIS 9 MILE TRIP, HE WAS ONLY $\frac{1}{4}$ OF THE WAY THERE - WITH $\frac{3}{4}$ TO GO. IF 4 GIRLS DIVIDED $9 EVENLY, EACH WOULD GET 25% OF THE TOTAL OR $2.25.

Tasks: *Pupils complete sentences and a chart involving percents and write stories to fit statements involving percents.*

Purpose: help pupils understand and use percent notation.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations; measurement.

The Lesson: After the pupils have completed the sentences in I, they ought to be able to give examples in which such sentences might arise in everyday life.

"I've already finished 50% of the 38 problems we were asked to complete. I've got 19 left to do."

"Eight people are absent today from Glee Club. That is 10% of 80 members."

The headlines selected in III were deliberately chosen as examples of what used to be singled out and called "three cases of percent." In terms that ought to be familiar by now, the first example is a case in which the factors are given and the product is missing; in the second, a factor is missing; and in the third, a factor is missing.

We know that when a factor is missing, we often rewrite the problem as a division problem; so headlines *b* and *c* can be rewritten:

$7 \div 10\% = \_\_\_\_\_$

$2.25 \div 9 = \_\_\_\_\_\%$

To solve these division problems, we can use the following procedures:

$10\% \overline{)7} \rightarrow .1 \overline{)7} \rightarrow .1 \times 10 \overline{)7 \times 10} \rightarrow 1 \overline{)70}$ (70)

$9 \overline{)2.25} \rightarrow 9 \overline{)2.25}$ (.25 → 25%)

I. Rearrange the numbers given in the green region in the order indicated by the signs.

Remember: $<$ is less than
$>$ is greater than
$=$ is equal to

a.	$\frac{1}{2}$ 75% .4 $\frac{8}{10}$ .49 41%	.4 < 41% < .49 < $\frac{1}{2}$ < 75% < $\frac{8}{10}$
b.	.7 125% 1 $\frac{16}{10}$ 85% 1.5	$\frac{16}{10}$ > 1.5 > 125% > 1 > 85% > .7
c.	$\frac{1}{2}$ .03 3 .3 30 $\frac{1}{4}$	.03 < $\frac{1}{4}$ < .3 < $\frac{1}{2}$ < 3 < 30
d.	1.48 $\frac{9}{10}$ 99% 150% $1\frac{1}{100}$ $1\frac{1}{2}$	150% = $1\frac{1}{2}$ > 1.48 > $1\frac{1}{100}$ > 99% > $\frac{9}{10}$
e.	$\frac{3}{10}$ 20% $\frac{1}{4}$ $\frac{3}{5}$ $\frac{1}{5}$ 60%	$\frac{3}{5}$ = 60% > $\frac{3}{10}$ > $\frac{1}{4}$ > $\frac{1}{5}$ = 20%
f.	1.00 $\frac{6}{3}$ 200% 125% 1.9 $\frac{10}{8}$	1.00 < $\frac{10}{8}$ = 125% < 1.9 < 200% = $\frac{6}{3}$
g.	39% $\frac{9}{20}$ $\frac{3}{100}$ .49 2% $\frac{2}{5}$	2% < $\frac{3}{100}$ < 39% < $\frac{2}{5}$ < $\frac{9}{20}$ < .49
h.	$\frac{1}{8}$ 1.0 6 $\frac{8}{2}$ $\frac{4}{36}$ 12%	6 > $\frac{8}{2}$ > 1.0 > $\frac{1}{8}$ > 12% > $\frac{4}{36}$
i.	$\frac{7}{20}$ .06 .35 6% $\frac{3}{50}$ 35.0%	$\frac{7}{20}$ = .35 = 35% > .06 = $\frac{3}{50}$ = 6%
j.	.75 1000% 50% 10 $\frac{15}{20}$ $\frac{11}{22}$	50% = $\frac{11}{22}$ < $\frac{15}{20}$ = .75 < 1000% = 10
k.	$\frac{11}{100}$ $\frac{7}{50}$ 13% $\frac{3}{25}$ .1 9%	9% < .1 < $\frac{11}{100}$ < $\frac{3}{25}$ < 13% < $\frac{7}{50}$
l.	$\frac{1}{20}$ $\frac{1}{100}$ $\frac{5}{2}$ 5% 3 250%	3 > 250% = $\frac{5}{2}$ > $\frac{1}{20}$ = 5% > $\frac{1}{100}$

II. Using $<$, $>$, or $=$, make three statements about the number given in the green region.

A.	25% x 7	> 1	< 2	> 1.5
B.	10% x 83	> 8.00	< 80.0	< 800
C.	1% x 97	< 10	< 1	> .7
D.	.10 x 25	> .20	> 2.0	< 20
E.	$\frac{10}{11}$ + $\frac{19}{20}$	> .5	< 2.5	> 1
F.	.20 x 37	> .6	> 7	< 18
G.	$\frac{1}{3}$ + $\frac{1}{2}$	> $\frac{1}{6}$	< 1	< 5
H.	$\frac{7}{10}$ − $\frac{30}{100}$	> $\frac{1}{7}$	= .4	< 2.1
I.	7 ÷ 2	< 34	> .4	< 4
J.	.7 ÷ 2	> .3	< 5	< 1.4

K.	3 x $\frac{11}{10}$	> 2	< 7	< 10
L.	3 x .2	< $\frac{6}{1}$	= $\frac{6}{10}$	> $\frac{6}{100}$
M.	.3 x .2	< $\frac{6}{1}$	< $\frac{6}{10}$	= $\frac{6}{100}$
N.	.5 x .7	< 35	< 3.5	= .35
O.	8 x .4	< 32	= 3.2	> .32
P.	1.2 x 1.2	> 1	< 2	< 14.4
Q.	1.0 x 9	> .90	> .09	= 9.0
R.	.3 x 25	= 7.5	> $\frac{3}{8}$	> 75%
S.	5 x .5	> 1%	> 10%	> 100%
T.	.8 x .5	< 4.0	> .04	= .4

105 Ordering decimal fractions and percents

Tasks: *Pupils rearrange numbers expressed in different notations and make statements about the sums, differences, products, and quotients of numbers expressed in different notations.*

Purpose: To test pupil understanding and mastery of fractions in common fraction, decimal fraction, and percent notations.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Pupil pages 105 and 106 are designed for use as general checkups with particular attention given to fractions in various notations. Exercise I is more difficult than it seems at first. Pupils ought to have scratch paper handy to experiment with an ordering that will fit the pattern set by the equality and inequality signs. Notice that each ordering moves from smallest to largest or from largest to smallest.

As preparation for this checkup pass out six slips of paper to each of the pupils, and tell them to write a different number on each slip—using percent notation on two, decimal fraction notation on two and common fraction notation on two. Collect the slips and redistribute them. Then tell the class that the first to put his slips in order from smallest to largest wins. Notations for the same number can be grouped together.

Perhaps one or two pupils might volunteer to arrange all the slips of paper in order along the chalk rail.

The activity in exercise II is closely related to that in exercise I. However, each example calls for a computation and then a comparison with each of three numbers.

Be sure that there is no question about the method of indicating the relationship. Consider similar examples, such as:

75% x 20	= 15	> 1.50	< 51
	15 = 15	15 > 1.50	15 < 51
.3+.07	> .10	< 1	> .037
	.37 > .10	.37 < 1	.37 > .037

including at least example A of exercise II:

25% x 7	> 1	< 2	> 1.5
	$1\frac{3}{4} > 1$	$1\frac{3}{4} < 2$	$1\frac{3}{4} > 1.5$

Watch the Signs! (Use scratch paper if you need to.)

I.

+	17	46	88
24	41	70	112
70	87	116	158
52	69	98	140
37	54	83	125

II.

X	7	4	9
6	42	24	54
3	21	12	27
8	56	32	72
5	35	20	45

III.

+	.5	2.5	5	.25
8	8.5	10.5	13	8.25
1.8	2.3	4.3	6.8	2.05
.18	.68	2.68	5.18	.43
.8	1.3	3.3	5.8	1.05

IV.

+	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{1}{2}$	1	$\frac{3}{4}$	$1\frac{1}{6}$	$1\frac{1}{4}$
$\frac{1}{3}$	$\frac{5}{6}$	$\frac{7}{12}$	1	$1\frac{1}{12}$
$\frac{2}{3}$	$1\frac{1}{6}$	$\frac{11}{12}$	$1\frac{1}{3}$	$1\frac{5}{12}$

V.

X	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{3}{8}$
$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{2}{9}$	$\frac{1}{4}$
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{4}{9}$	$\frac{1}{2}$

VI.

+	$3\frac{1}{2}$	$1\frac{5}{6}$	$4\frac{3}{8}$	$\frac{3}{10}$
$4\frac{1}{3}$	$7\frac{5}{6}$	$6\frac{1}{6}$	$8\frac{17}{24}$	$4\frac{19}{30}$
$2\frac{1}{2}$	6	$4\frac{1}{3}$	$6\frac{7}{8}$	$2\frac{4}{5}$
$\frac{2}{10}$	$3\frac{7}{10}$	$2\frac{1}{30}$	$4\frac{23}{40}$	$\frac{1}{2}$

VII.

−	50	84	57
28	22	56	29
13	37	71	44
49	1	35	8
37	13	47	20

VIII.

X	3	.2	.08
1.2	3.6	.24	.096
.5	1.5	.10	.040
14	42	2.8	1.12
.9	2.7	.18	.072

IX.

X	10	.19	4.1
3	30	.57	12.3
.2	2.0	.038	.82
.005	.05	.00095	.0205
.7	7.0	.133	2.87

X. Mr. Boles sold floor tile. Some of the tiles were 1 foot square — 12 inches by 12 inches — and were sold only in packs of 1 dozen each. To save time, Mr. Boles made a chart so that he could quickly answer questions as soon as he knew the length and width of the floor to be covered with tile. Fill in the blanks in the part of his chart shown below.

(In the chart below, all rooms are more than 4 feet in length and in width.) All lengths and widths are given to the nearest foot.

	Length of the room	Width of the room	No. of tiles needed	No. of packages needed	No. of tiles not used
a.	7 ft.	5 ft.	35	3	1
b.	7 ft.	6 ft.	42	4	6
c.	7 ft.	7 ft.	49	5	11
d.	7 ft.	8 ft.	56	5	4
e.	7 ft.	9 ft.	63	6	9

	Length of the room	Width of the room	No. of tiles needed	No. of packages needed	No. of tiles not used
f.	11	5	55	5	5
g.	15	5	75	7	9
h.	15	15	225	19	3
i.	14	16	224	19	4
j.	13	7	91	8	5

Tasks: *Pupils complete the grids in exercises I-IX and the chart in exercise X.*

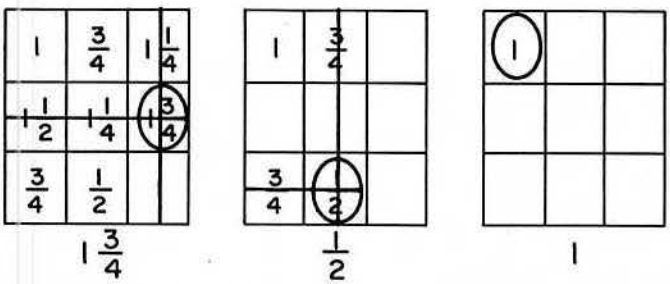
Purpose: To test pupil mastery of addition and multiplication of whole numbers and fractions.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: This activity has a rather unexpected twist. Ask a volunteer to point to any of the nine entries in this 3-by-3 addition table.

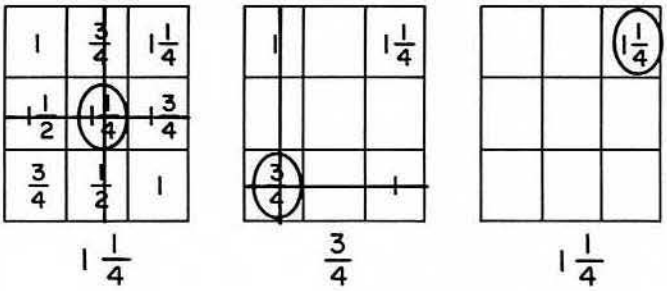
+	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$
$\frac{1}{2}$	1	$\frac{3}{4}$	$1\frac{1}{4}$
1	$1\frac{1}{2}$	$1\frac{1}{4}$	$1\frac{3}{4}$
$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	1

Note his selection and cross out that row and column. Have one of the remaining numbers selected. Cross out the row and column in which it appears. A single number is left. Finally, find the sum of the three numbers selected by this process. Here is one possible outcome:



$$1\frac{3}{4} + \frac{1}{2} + 1 = 3\frac{1}{4}$$

Try again.

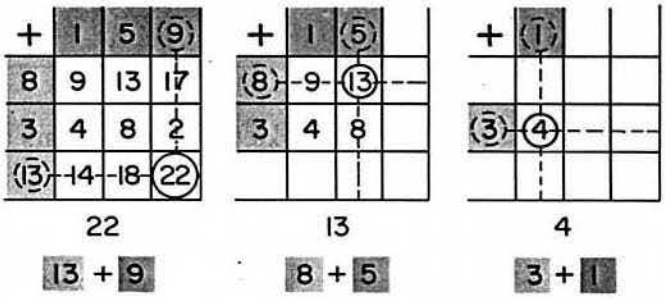


$$1\frac{1}{4} + \frac{3}{4} + 1\frac{1}{4} = 3\frac{1}{4}$$

No matter what selections are made, the sum will always be 3¼. Try another—perhaps with more items, such as a 4-by-4 table. As long as there are the same number of rows and columns, the final sum will always be the sum of the generating numbers, the numbers across the top and down the left edge. In the example above:

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{3}{4}\right) + \left(\frac{1}{2} + 1 + \frac{1}{4}\right) = 3\frac{1}{4}$$

Here is another example:



$$22 + 13 + 4 = 39$$

$$(13 + 9) + (8 + 5) + (3 + 1) = 39$$

$$(1 + 5 + 9) + (8 + 3 + 13) = 39$$

Experiment with a multiplication table and multiply the entries selected in the same manner—to learn that the product will be the same as the product of the generating numbers.

I
GOAL
10
20
30
40
50
40
30
20
10
GOAL
Complete these four systems of labeling.
II
C
F
120
110
100
90
80
70
60
50
40
30
20
10
0
10
20
30
40
50
60
240
220
200
180
160
140
120
100
80
60
40
20
0
20
40
60
80
III
Jack
Jill
5TH N
4TH N
Art
3rd N
School
CENTER N
Dick
Library
Mary
2nd N
Ed
Phil
Betty
1st N
Tom
Hardware Store
Monument
Bank
MAIN E
MAIN W
City Hall
1st S
Bill
Jane
Theatre
2ND S
3RD W
2ND W
1st W
CENTER S
1st E
Helen
2nd E
3rd E
3RD S
Bob
4TH S
IV
PARKING
4B
3B
2B
1B
ENTER
1A
2A
3A
4A
5
NO PARKING

107 Exploring directed numbers

Tasks: *Pupils complete four systems of labeling.*

Purpose: To help pupils understand ways of labeling both "halves" of a line as a basis for labeling an extended number line.

Unifying Ideas: Structure; number and counting; numeration; functions and relations; geometry; measurement.

The Lesson: Four methods of labeling are shown on pupil page 107. They have one important characteristic in common. Each has a recognizable point, an *origin*, from which the labeling proceeds in two or more directions—the 50-yard line marker on a football field, the entrance in the parking lot shown, the 0 on the thermometer, and the monument on the map.

The parking lot and map labels consist of two elements: a number indicating distance from the origin, and a letter indicating direction from the origin. Direction is shown on the thermometer by the use of color—everyday usage tells us that blue is for above 0° and white is for below 0°. On the football field, the spectators need to know which team is defending which goal—and that it changes each quarter of the game.

After discussion of these systems of labeling, questions are in order:

How far is it between the two 40-yard markers?

How many degrees must the temperature fall from its present value before the thermometer shows 5° below zero?

How far is the bank from the hardware store?

How far is it from the corner of 3rd N and 2nd E to the corner of 2nd S and 2nd E?

What will the thermometer read if it rises 10° above its present reading and then falls 30°?

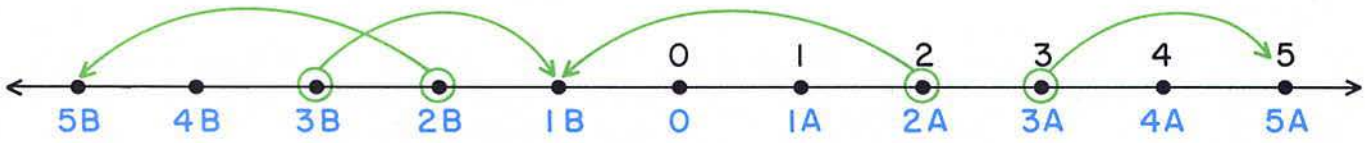
Put a line on the board and label the points:



If we consider the distance between A and K as a unit, how far is it from K to C? D to M? A to R? C to L?

Suppose that Bill is at K, and Jim is at R. How far apart are they? Each takes a trip 1 unit in length. Now how far apart are they? (To answer this question you need to know not only the distances for the trips but also the directions for the trips.)

The parking lot system and both scales on the thermometer suggest an extension of the number line.

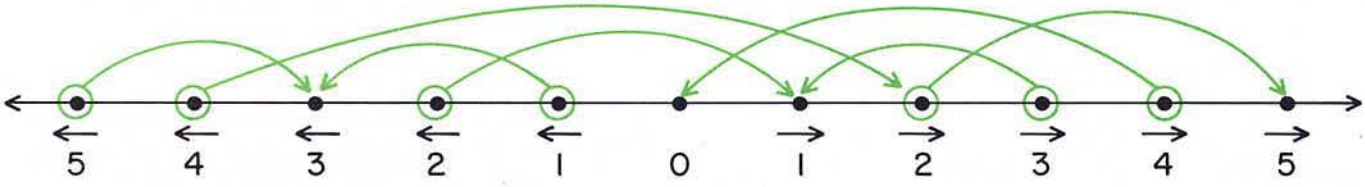


If you start at 2B and jump 3 spaces to the left, you land on 5B.
(Read that larger arrow as "goes to.")

2B, ← 3 → 5B 3B, → 2 → 1B

2A, ← 3 → 1B 3A, → 2 → 5A

Let's mark the number line with labels that indicate the way you would get there if you started at 0.



Start at 0. Take a jump of 5 to the left, then take a jump of 2 to the right. You land at a point 3 to the left of 0.

← 5, → 2 → ← 3

← 1, ← 2 → ← 3 → 2, → 3 → → 1

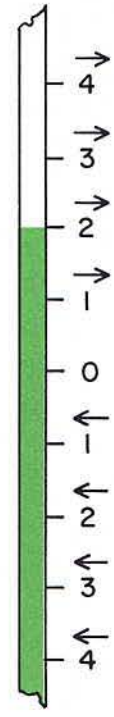
→ 4, ← 4 → 0 → 2, → 3 → → 5

→ 3, ← 2 → → 1 ← 4, → 6 → → 2

First a jump of . . .

and then a jump of . . .	← 3	← 2	← 1	0	→ 1	→ 2	→ 3
→ 3	0	→ 1	→ 2	→ 3	→ 4	→ 5	→ 6
→ 2	← 1	0	→ 1	→ 2	→ 3	→ 4	→ 5
→ 1	← 2	← 1	0	→ 1	→ 2	→ 3	→ 4
0	← 3	← 2	← 1	0	→ 1	→ 2	→ 3
← 1	← 4	← 3	← 2	← 1	0	→ 1	→ 2
← 2	← 5	← 4	← 3	← 2	← 1	0	→ 1
← 3	← 6	← 5	← 4	← 3	← 2	← 1	0

On the thermometer shown here, "→" means above 0 and "←" means below 0.



If the thermometer read . . .

and then . . .	→ 5	← 8	← 21	→ 72	→ 60
rose 9°	→ 14	→ 1	← 12	→ 81	→ 69
fell 15°	← 10	← 23	← 36	→ 57	→ 45
fell 11°	← 6	← 19	← 32	→ 61	→ 49
rose 28°	→ 33	→ 20	→ 7	→ 100	→ 88
fell 16°	← 11	← 24	← 37	→ 56	→ 44

Tasks: *At the top of the page, pupils complete sentences using a number line to test the truth of such statements; at the bottom of the page, pupils complete the two grids.*

Purpose:
bers using number line models.

Unifying Ideas: Structure; number and counting; numeration; addition and subtraction; functions and relations; geometry; measurement.

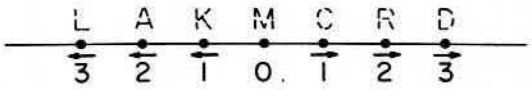
The Lesson: Copy this chart and drawing on the board and ask: What might this shorthand mean?

	Bill starts	Jim starts	Distance apart
a	$\overleftarrow{1}$	$\overrightarrow{1}$	5
b	$\overrightarrow{1}$	$\overrightarrow{1}$	3
c	$\overrightarrow{1}$	$\overleftarrow{1}$	1
d	$\overleftarrow{1}$	$\overleftarrow{1}$	3



Row *a* might mean that Bill started at *K* and went 1 unit to the left, and that Jim started at *R* and went 1 unit to the right. After these trips they were 5 units apart. Or, it might tell us that Bill started at *A* and went 1 unit to the left; after these trips they were 5 units apart. Etc.

Suppose that we replace the letter labels on the line with a shorthand something like that on the chart of trips. Let's assign 0 to one point and label the others to indicate what kind of trip you would take to get there from 0.



How would you get from $\overleftarrow{2}$ ("left arrow 2") to $\overrightarrow{1}$ ("right arrow 1")? Answer: Take a trip of 3 to the right; in other words, take a $\overrightarrow{3}$-trip.

Suppose that each of the following trips (I, II, and III) starts at 0. Each is a combination of two part-trips. (A 2 with a left-pointing arrow above it will indicate a trip of 2 units to the left, and a 2 with right-pointing arrow above it will indicate a trip of 2 units to the right.) Trip I starts at 0. The first part-trip goes 3 units to the left; the second part-trip goes 2 units to the right from the end point of the first. Where does trip I end? Answer: at $\overleftarrow{1}$.

TRIP	First part	Second part	Ends at
I	$\overleftarrow{3}$	$\overrightarrow{2}$	$\overleftarrow{1}$
II	$\overleftarrow{2}$	$\overrightarrow{1}$	$\overrightarrow{3}$
III	$\overrightarrow{2}$	$\overleftarrow{4}$	$\overleftarrow{2}$

Without any further explanation, let pupils work through page 108 independently.

A possible source of initial confusion to pupils is the common use of a notation such as "$\overleftarrow{2}$" both to denote *the measure of a trip* of length 2 in a certain direction and to serve as a label for *the point* 2 units in that direction from an origin on a scaled line. A thoughtful pupil might wonder how a trip-measure could also be a point. Of course, a trip-measure is *not* a point. The difficulty resides in the fact that the notation can be used in two different ways, and is often so used in the same discussion.

In the "headlines" below, arrows are used to indicate directions. They can be used for many kinds of directions. For example, if → means *forward* then ← will mean *backward*. We say that the direction → is the opposite of the direction ←, and that ← is the opposite of →. *(Fill in the missing directions.)*

If → means . . .	Up	Profit	EAST	ABOVE	Found	DAY
then ← means . . .	Down	LOSS	West	Below	LOST	NIGHT

Write a story for each headline, and write a headline for each story.

a $\overrightarrow{17}, \overleftarrow{25} \rightarrow \overleftarrow{8}$ *Mr. Jones drove 17 miles north from his home on Route 7. Then he turned around and drove 25 miles south. Where was he then?*

b $\overleftarrow{8}, \overrightarrow{29} \rightarrow \overrightarrow{21}$ JAN LOST 8 OF THE 29 FOUR-LEAF CLOVERS SHE HAD FOUND. SHE THEN HAD 21 LEFT.

c $\overleftarrow{1.25}, \overleftarrow{3.75} \rightarrow \overleftarrow{5}$ BILL DRILLED A HOLE $1\frac{1}{4}$ INCHES DEEP. IT NEEDS TO BE 5 INCHES DEEP. THAT MEANS HE HAS TO DRILL THE HOLE $3\frac{3}{4}$ INCHES DEEPER.

d $\overrightarrow{2\frac{1}{2}}, \overleftarrow{4} \rightarrow \overleftarrow{1\frac{1}{2}}$ AFTER THE TEMPERATURE WENT DOWN 4 DEGREES, IT READ $1\frac{1}{2}$ DEGREES BELOW ZERO.

e $\overrightarrow{5}, \overleftarrow{13}, \overrightarrow{15} \rightarrow \overrightarrow{7}$ THE ELEVATOR WENT UP 5 FLOORS, DOWN 13, AND THEN UP 15 FLOORS. IT WAS 7 FLOORS ABOVE THE POINT AT WHICH IT STARTED.

f $\overleftarrow{8}, \overrightarrow{17} \rightarrow \overrightarrow{9}$ Bill started down the basement stairs. After going down 8 stairs, he changed his mind. Then he went up 17 stairs.

g $\overleftarrow{1\frac{1}{2}}, \overleftarrow{3} \rightarrow \overleftarrow{4\frac{1}{2}}$ Harry lost $1\frac{1}{2}$ pounds one week and twice as much the next. How had his weight changed?

h $\overleftarrow{63}, \overrightarrow{25} \rightarrow \overleftarrow{38}$ The skin diver went down 63 feet. After he came up 25 feet, he saw a beautiful piece of coral. How far was the coral beneath the surface?

i $\overleftarrow{5}, \overrightarrow{32} \rightarrow \overrightarrow{27}$ Mr. Swanson found he had gone 5 miles in the wrong direction. "Now I'm 32 miles from home." How far would he have had to go if he had not made a mistake?

j $\overleftarrow{175}, \overrightarrow{210} \rightarrow \overrightarrow{35}$ Bill walked 175 feet into a cave. Then he turned around and walked 210 feet. Where was he?

k $\overrightarrow{212}, \overleftarrow{75} \rightarrow \overrightarrow{137}$ The water had cooled 75 degrees Fahrenheit after it had boiled. This happened at the seashore last summer.

109 Writing headlines for stories and vice versa

Tasks: *Pupils complete pairs of opposites, write stories suggested by open sentences, and write mathematical sentences suggested by stories.*

Purpose: To help pupils understand directed numbers as measures of "trips" made in one of two opposite directions.

Unifying Ideas: Structure; numeration; addition and subtraction; measurement.

The Lesson: Pupil page 109 explores ways in which these "right arrow" and "left arrow" numbers reflect patterns in the world of events. Several interpretations are given:

up and down	above and below
profit and loss	found and lost
east and west	

Urge pupils to find others:

north and south	in and out
future and past	high and low

In each case, several examples involving numbers should be found. Examples:

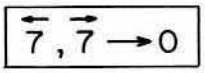
"I went up 13 stairs and down 3. I was 10 steps up from my starting point."

"My birthday was a week ago and my sister's is the day after tomorrow. Our birthdays are 9 days apart."

"The point of a stake is 10 inches in the ground. The stake is 2 feet long. The top of the stake is 14 inches above the ground."

"The temperature fell from 22° above zero to 19° above zero. This is a fall of 3°."

Next, use a headline story technique to strengthen understanding of ways in which these numbers help express familiar patterns:

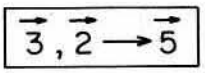


"I walked 7 blocks west to my grandmother's, and then walked 7 blocks east—which brought me back home."

"Bill dove down 7 feet in the water and came back up to the surface."

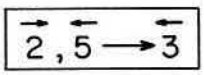
"I went into the cave 7 feet and back out."

"The temperature fell 7° and then rose 7°."



"A trip of 3 miles to the east and then 2 miles more to the east is the same as a 5-mile trip to the east."

"I piled 2 feet of dirt on top of a pile that was already 3 feet high. It was then 5 feet high."



"Harry gained 2 pounds and lost 5 pounds. He weighs 3 pounds less than before."

"Our team gained 2 yards on one play and lost 5 yards on the next. That's a net loss of 3 yards."

After considerable class participation in creating such examples and discussing them, most of the pupils can work successfully by themselves on pupil page 109.

"Directed numbers" is one of the names that has been given to those numbers that can be used to label points on the number line on both sides of 0. Where do you think that name came from?

Others call them "signed numbers." Why might that be an appropriate name?

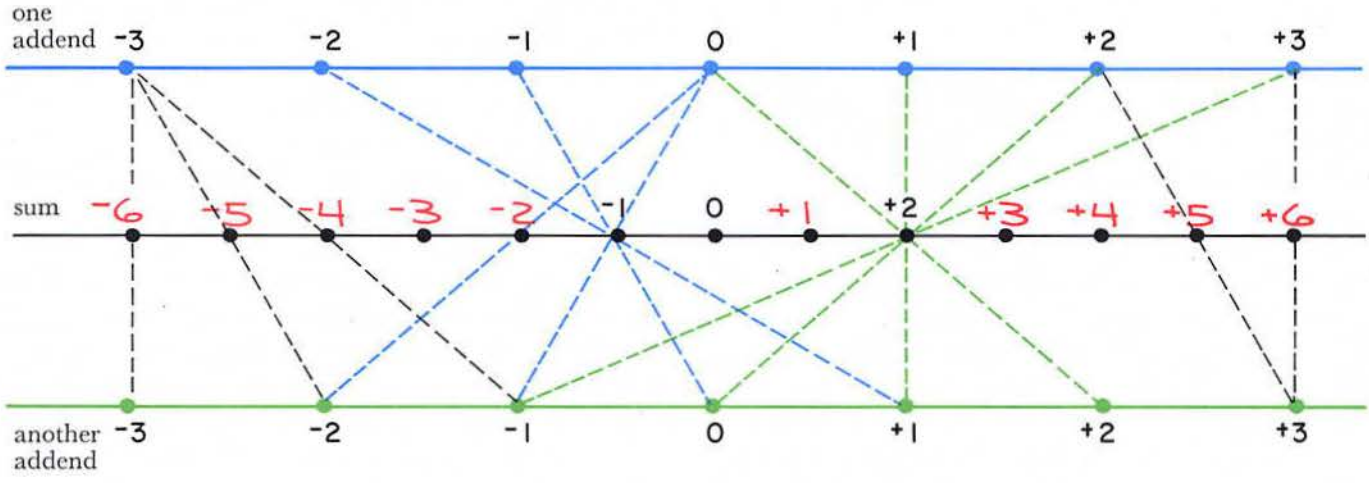
Most mathematicians do not use arrows such as we have used . . . $\overrightarrow{3}$ or $\overleftarrow{4}$. They prefer to indicate a direction from 0 in the following way:

+3 . . . read "positive three," and
−4 . . . read "negative four."

We shall use these customary symbols.

What do you suppose we could mean by
ADDITION of directed numbers?

Here is a picture using the idea of doing addition with a number ladder:



Can you finish labeling the middle line?

Complete the following sentences.

1. (+1) + (+1) = +2
2. (0) + (+2) = +2
3. (+2) + (0) = +2
4. (+3) + (−1) = +2
5. (−2) + (+1) = −1
6. (−1) + (0) = −1
7. (0) + (−1) = −1
8. (+1) + (−2) = −1
9. (−3) + (−1) = −4
10. (−3) + (−2) = −5
11. (−3) + (−3) = −6
12. +1 = (−1) + (+2)
13. −2 = (−3) + (+1)
14. (−3) + (+3) = 0
15. 0 = (−1) + (+1)
16. (+3) + (+2) = +5
17. (−2) + (+4) = +2
18. −5 = (−1) + (−4)
19. −15 = (−7) + (−8)
20. (−6) + (+15) = +9
21. +6 = (+15) + (−9)
22. (−10) + (+3) = −7
23. (+3) + (−2) = (−2) + (+3)
24. (−2) + (−3) + (+4) = −1
25. (+4) + (−2) + (−3) = −1
26. (+2) + (−7) = (−3) + (−2)
27. (+9) + (−5) = (−5) + (+9)
28. (−4) + (−4) + (−4) = −12
29. (+5) + (+5) + (+5) = +15
30. (−17) + (−8) + (+17) = −8
31. (−25) + (+9) + (+16) = 0
32. +3 = (−7) + (+11) + (−1)
33. (−2) + (+10) + (−12) = −4

110 Adding directed numbers by means of a number ladder

Tasks: *Pupils label a number ladder (or "adding machine") and complete the sentences (1-33), using the number ladder if necessary.*

Purpose: To introduce the vocabulary of and mathematical symbols for "directed numbers" and "signed numbers." To provide experience in addition and subtraction of such numbers.

Unifying Ideas: Structure; number and counting; addition and subtraction; functions and relations.

The Lesson: Although we introduce the accepted convention of "+3" (read as "positive three") and "−4" (read as "negative four"), we point out that there is no change at all in what these symbols stand for.

All examples thus far have had a certain common aspect—a pair of "trips" or directed changes. The idea running through them all we call *addition*. With that idea in mind, we write:

$$(^{+}1) + (^{+}1) = {}^{+}2$$

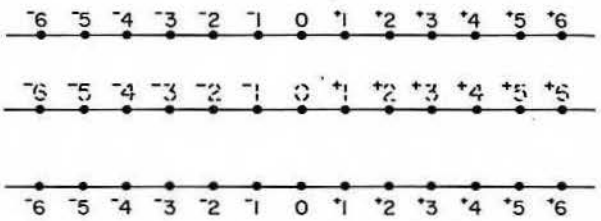
"Positive one plus positive one equals positive two."

$$(^{-}1) + (^{+}3) = {}^{+}2$$

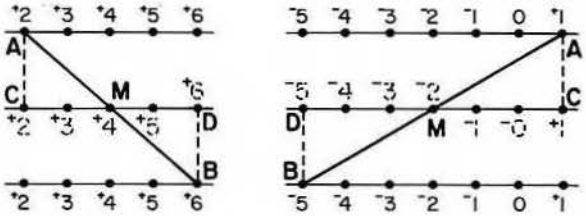
"Negative one plus positive three equals positive two."

We use parentheses to avoid confusion between the signs for positive and negative and the signs for the operations of addition and subtraction.

Pupil page 110 presents a "number ladder," *a nomogram* for finding sums of directed numbers. Here is how to make one. Start with three parallel number lines all with the same scale and arranged so that the 0's are aligned and one of the lines is midway between the others:

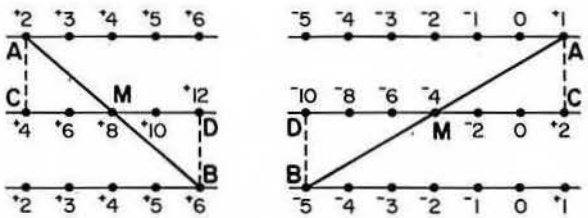


Select a pair of addends, one for each of the first and third number lines. If the segment is drawn between these points, it will cross the middle line at the *average* of the given addends.



In each example, M is halfway between C and D (since triangles AMC and BMD are congruent): $^{+}4$ is halfway between (or is the average of) $^{+}2$ and $^{+}6$, and -2 is the average of $^{+}1$ and -5. So this procedure does indeed find the average of two directed numbers.

But, the average of two directed numbers is half their sum. So, their sum is twice the average. Hence, to find sums you can first find the average and then double it. Or, you can simply rescale the middle line and find sums directly:



Of course, the pupils can discover and check this procedure by computing a few sums in the way described in Notes on Pupil Page 109:

$$^{+}3, {}^{-}1 \longrightarrow {}^{+}2$$

Think of a Number

"I once played a game that went like this," Harry said.

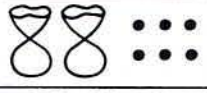
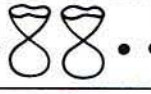
"Think of a number. Don't tell me, but write it down so you won't forget.

"Add 3. Double your result. Subtract 4. Divide by 2. Subtract the number you thought of first. If your answer isn't 1, then you've made a mistake!"

Most of the class was amazed — except one or two who remembered the trick. The class made a chart of what had happened.

(Please complete the record.)

	Al	Bob	Don	Meg	Art	Sam	Alice	Jim	Nancy	Paul	Helen
Think of a number.	5	7	0	3	25	31	19	27	105	158	589
Add 3.	8	10	3	6	28	34	22	30	108	161	592
Double the result.	16	20	6	12	56	68	44	60	216	322	1,184
Subtract 4.	12	16	2	8	52	64	40	56	212	318	1,180
Divide by 2.	6	8	1	4	26	32	20	28	106	159	590
Subtract the number you thought of first.	1	1	1	1	1	1	1	1	1	1	1

The picture below suggests the way this trick works.		A convenient shorthand
Think of a number. 	THINK OF A NUMBER. Imagine that you put that many counters in a bag. Don't tell anyone how many.	
Add 3. 	ADD THREE. Put 3 counters by the bag. Now you have a bag and 3 counters more.	 + 3
Double it. 	DOUBLE IT. You need another bag like the first, and another 3 counters. That's 2 bags and 6 counters.	2x + 6
Subtract 4. 	SUBTRACT FOUR. That leaves 2 bags and 2 counters.	2x + 2
Divide by 2. 	DIVIDE BY TWO. Take away one bag and one counter, leaving one bag and one counter.	x + 1
Subtract the number you thought of first. 	SUBTRACT THE NUMBER YOU THOUGHT OF FIRST. No matter how many counters you had in it. One left!	1

Remarks about the shorthand.

In the shorthand, we erase the top and bottom of the bag, and the remainder looks much like the letter x. Of course, any letter would do to hold a place for "the number you thought of first" — n or y — you choose it.

Instead of drawing counters, we used numbers to keep track of "how many."

Instead of drawing two bags, we shortened that to $2x$. In other games we may wish to show 100 bags, and $100x$ is a short way of indicating that.

111-128 Introducing algebraic and geometric ideas through such approaches as (1) Think-of-a-Number games, (2) looking for shortcuts in multiplication, (3) Giants-and-Dwarfs arithmetic, (4) diagraming What's-My-Rule? games, and (5) space mathematics.

Tasks: *Pupils study a* Think-of-a-Number *game and complete an analysis of it. They study shorthand for the steps taken in the game.*

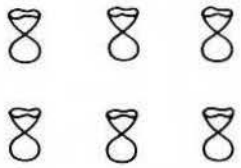
Purpose: To help pupils explore the steps of a game by using physical representations and algebraic shorthand for these steps.

Unifying Ideas: Structure; number and counting; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: The transition from arithmetic to algebra should be imperceptible. The pupil should not be able to say, "At this page I left arithmetic and started algebra," because these subjects are essentially one.

When a pupil says, "If you take any number, multiply it by 3, and double the result, the answer will be 6 times the number," he is making a commonsense statement about arithmetic. The statement is commonly regarded as algebraic, because it is a *general* statement about numbers. The statement may be formulated in a special shorthand, but the shorthand is only a device. The important aspect of algebra is that it deals with general statements. The statement "If you double 3 times any number, you get 6 times that number" is a very small step away from the statement "For each x, if you double $3x$, you get $6x$."

It is also possible to illustrate the above statement in the following way. We draw 3 bags. We say that each bag has the same number of counters in it. We then draw 3 more bags. These, too, each contain the same number of counters.



We can say we have "three bags twice" or "six bags." The picture demonstrates that "three times a number, doubled" and "six times a number" are two ways of saying the same thing.

In the columns headed *Bob*, *Don*, *Meg*, *Art*, *Paul*, and *Helen*, pupils are given the numbers thought of and are asked to carry out the steps of the game. Pupils will observe that the answer is 1 in each case. But in the columns headed *Sam*, *Alice*, *Jim*, and *Nancy*, pupils are only given numbers that occur during the steps involved in the trick. They are not given the numbers thought of. *It is perfectly satisfactory for pupils to complete the trick by any method they choose.* Most pupils will probably work backwards: "After adding 3 to his number, Sam got 34; he must have started with 31." "Alice had 44 after doubling; so, she must have had 22 before doubling; before adding 3 she must have had 19." In the same way pupils can work out Jim's and Nancy's columns.

We shall use x to hold a place for any number you think of. We could use any mark or letter, but on this page we shall use x.

In the spaces that follow, choose your answer from the expressions in the blue-tinted block at the top of the next column.

Choose the expression that best fits the description given. The first two blanks are filled in to show what is wanted.

a. any number you may think of — x

b. one more than the number you may have thought of — x+1

c. one less than the number you thought of — x-1

d. two less than the number you thought of — x-2

e. three more than the number you thought of — x+3

f. three times the number you thought of — 3x

g. two more than the number you thought of — x+2

x $x+1$ $x+2$ $x+3$ $2x$

$x-1$ $x-2$ $3x$ x^2 $x1$

(x^2 is shorthand for x times x — "x squared.")

h. the number you thought of multiplied by two — 2x

i. the number you thought of multiplied by itself — x²

j. twice the number you thought of — 2x

k. the number you thought of with one subtracted — x-1

l. the number you thought of with two added — x+2

m. three added to the number you thought of — x+3

n. three multiplied by the number you thought of — 3x

o. the square of the number you thought of — x²

p. the number you thought of multiplied by one — x1

"Think of another number," Harry broke in.

"Then follow these directions: first double it. Next, add 9. Then add the number you thought of first. Divide by 3. Double this result. Subtract 6 and tell me your answer. I'll tell you the number you thought of first."

	Shorthand	Al	Bob	Don	Meg	Art	Helen	Alice
Think of a number.	x	4	2	1	0	50	75	127
Double it.	2x	8	4	2	0	100	150	254
Add 9.	2x +9	17	13	11	9	109	159	263
Add the number you thought of first.	3x +9	21	15	12	9	159	234	390
Divide by 3.	x +3	7	5	4	3	53	78	130
Double this result.	2x+6	14	10	8	6	106	156	260
Subtract 6.	2x	8	4	2	0	100	150	254
The number you thought of is	x	4	2	1	0	50	75	127

Tasks: *Pupils match algebraic expressions with verbal descriptions. They complete and reconstruct a* Think-of-a-Number *game.*

Purpose: To provide practice in using algebraic expressions.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Pupils may wish to give the answer $1+x$ for "one more than the number thought of," where we have given $x+1$. Both $1+x$ and $x+1$ are correct (even though $1+x$ does not appear in the blue-tinted block). $3+x$ and $x+3$ tell you the same thing (as long as you believe in commutativity of addition); and $2+x$ tells you the same thing as $x+2$. On the other hand, $2-x$ does not tell you the same thing as $x-2$. The first indicates the result of subtracting your number from 2, while $x-2$ indicates the result of subtracting 2 from your number.

It usually takes time for students to get used to the fact that in algebra multiplication signs such as $\times$ and $\cdot$ are usually omitted. There are several ways of dealing with this. It is possible to point out that it is natural for us to say "three sevens" when we mean three times seven.

We can also provide a transition. We can start with writing a multiplication sign, $3 \times x$, add a dot in the middle of the sign, $3 \times x$, and then agree to keep the dot only, $3 \cdot x$, thereby employing another standard form. Finally, we can omit the dot, omit the space, and reach the more usual form: $3x$. (At this stage, we accept $x \cdot 3$ or $x3$ as perfectly good equivalents of $3 \cdot x$ or $3x$.)

In exercise *p*, we give $x1$ as the answer. However, x must also be accepted as perfectly correct. The expressions $x1$ and x are equivalent.

Pupils usually need to be reminded many times *not* to read or think of $3x$ as "3 and x." Pictures may help. How do we picture x? As a bag. How do we picture $3x$? As three bags. Do these bags represent 3 counters *added* to the counters in a bag? No. If there were 8 counters in each bag, three bags would represent 24 counters, or 3 *times* 8.

Two exercises (*i* and *o*) involve x^2, "the square of the number you thought of." Pupils frequently act as if 10^2 meant twice 10. One should not allow pupils to embark on the algebra of x^2 and x^3 and x^5 until they are perfectly clear that 2^4 is $2 \times 2 \times 2 \times 2$, not 2×4; and that 2^3 is 8, not 6.

The work at the bottom of the page may be easier if pictures of bags are used. For example, in the fourth row we have $3x+9$, which we may picture as 3 bags and 9 counters. The next instruction is, "Divide by 3." If 3 bags and 9 counters are to be divided equally among 3 children, how much would each child get? Answer: a bag and 3 counters; shorthand: $x+3$. In the same way, the other steps can be worked out with the help of pictures.

In the row before the last, we expect a $2x$ in the *Shorthand* column. This means that the trick at this stage always give *twice* the number first thought of. So, to tell the original number, we divide by 2.

Notice that the last row gives x, the number thought of. Don has 1 in the last row—so, he began with 1, and we can work out his column from this. Meg has 0 in the $2x$ row. If twice her number is 0, her number must be 0. Art has 100 in the $2x$ row, and so his number must have been 50.

Each series of expressions include some of the different ways of saying the same thing. The usually preferred form is printed on a green background.

A. two times three . . . three times two . . . 2 multipled by 3 . . . 2×3 . . . $2 \cdot 3$ . . . $3 \cdot 2$. (The multiplication sign looks so much like the letter x we often use a dot in its place.)

B. the number you thought of times 4 . . . $x \cdot 4$ . . . $4 \cdot x$ . . . $4x$. (Where no confusion will result, we omit the dot. Why shouldn't we omit the dot in $2 \cdot 5$?)

C. twice the sum of five and nine . . . $2 \times (5 + 9)$ . . . $2 \cdot (5 + 9)$ . . . $2(5 + 9)$. (We can omit the dot here because no confusion results.)

D. the number you thought of . . . x . . . a . . . n . . . y . . . z . . . b . . . etc. (Any letter is as good as another. Be sure each time everyone understands what you mean.)

An agreement about the order of work to be done.

To prevent confusion, mathematicians agree that all indicated multiplications and divisions should be carried out first. Only then should indicated additions and subtractions be carried out. (If some different order is to be followed, parentheses are placed around the part of the expression to be dealt with first.)

Without this agreement, we could not tell which of the following was indicated:

Which one?

$2 + 4 \cdot 3 = 6 \cdot 3 = 18$

$2 + 4 \cdot 3 = 2 + 12 = 14$

Certainly 18 and 14 are very different results. But, because of our agreement about order of work, we know that the indicated multiplication should be carried out first.

If it were intended that the addition should be done first, that would have been indicated by parentheses. It would be:

$(2 + 4)3 = 6 \cdot 3 = 18$

With this agreement in mind, complete the mathematical sentences — *a* through *l* — in the column at the right.

a. $2 \cdot 3 + 1 =$ 7

b. $2 + 3 \cdot 4 =$ 14

c. $3 \cdot 2 - 1 =$ 5

d. $1 + 2 \cdot 3 =$ 7

e. $1 + 3 \cdot 2 =$ 7

f. $(2 + 3) \cdot 4 =$ 20

g. $2 \cdot (3 - 1) =$ 4

h. $4(3 + 2) =$ 20

i. $(3 - 1)5 =$ 10

j. $3(2 - 1) =$ 3

k. $(1 + 2)3 =$ 9

l. $(1 + 3)2 =$ 8

In the lines below, is the expression printed in green equivalent to the expression printed in blue?

			Are they equivalent?
1.	$1 + 2 \cdot 3$	$1 + 3 \cdot 2$	Yes
2.	$(1 + 2)3$	$(1 + 3)2$	No
3.	$2 \cdot 3 + 1$	$2(3 + 1)$	No
4.	$(2 + 3)4$	$2 \cdot 4 + 3 \cdot 4$	Yes
5.	$2(3 + 4)$	$(3 + 4)2$	Yes
6.	$3(2 + 4)$	$3 \cdot 2 + 3 \cdot 4$	Yes
7.	$2 + 3 \cdot 4 + 5$	$(2 + 3) \cdot (4 + 5)$	No
8.	$3x + 2$	$3(x + 2)$	No
9.	$2(x + 7)$	$2x + 14$	Yes

Notes on Pupil Page 113

Tasks: *Pupils study expressions for saying the same thing—"equivalent expressions." They study agreements regarding the order of performing operations, and carry out computations based on these agreements. They compare expressions to see if they are equivalent.*

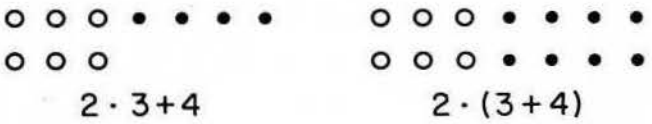
Purpose: To provide further experience in understanding and using algebraic shorthand and in using agreements regarding order of operations.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

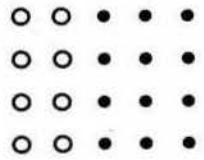
The Lesson: We must not omit the dot (for times) that appears between the 2 and the 5, because the result would be read as twenty-five (see the remark following B). In algebra, when we put two symbols such as 2 and x together to make $2x$, we wish the notation to mean multiplication. In arithmetic, when we put 2 and 5 together in 25, we wish the notation to mean something quite different—2 tens and 5. And we do not wish to give up either interpretation! We want the symbol 25 for twenty-five and we want the symbol $2x$ for "two times the number thought of."

The rest of this page deals with parentheses. What does a waiter mean when he orders two eggs and bacon from the kitchen? Does he mean that one customer wants bacon with two eggs, or that each of two customers wants eggs and bacon (or, that one customer is to get two eggs and another to get bacon)? We have a similar difficulty in arithmetic with "twice three and four." Does this mean twice seven or does it mean six and four? We can use parentheses to make the distinction; thus, $2\cdot(3+4)$ means twice the sum of three and four, while $(2\cdot3)+4$ means two threes and a single four. By agreement, if no parentheses are used, as in $2\cdot3+4$, we shall mean the latter.

Pictures of the following type may be helpful.



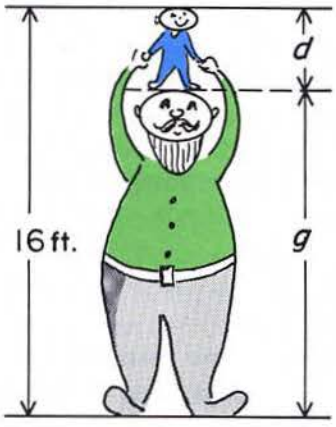
Ask pupils to draw pictures to illustrate $3\cdot5+4$ and $3\cdot(4+4)$. You might also draw pictures such as:



and ask pupils to supply the label. Here, $4\cdot(2+3)$ is correct and $4\cdot2+3$ is wrong. (There are other correct answers, such as $4\cdot5$ or $4\cdot2+4\cdot3$.)

Algebraic expressions are involved in the last two questions on this page. Is the expression $3x+2$ equivalent to the expression $3(x+2)$? Draw pictures. $3x+2$ corresponds to the sum of 3 bags and 2 counters. $3(x+2)$ corresponds to multiplying 3 by the sum of a bag and 2 counters. Doing so would result in the sum of 3 bags and 6 counters.

In the last question, $2(x+7)$ corresponds to doubling the sum of a bag and 7 counters. This would result in the sum of 2 bags and 14 counters.



I. A dwarf stands on a giant's head. Together, they are 16 feet tall. What is the height of the dwarf? What is the height of the giant?

Obviously, we don't have enough information to answer the questions. However, we can fill in the chart below:

If the height of the giant and dwarf together is 16 feet

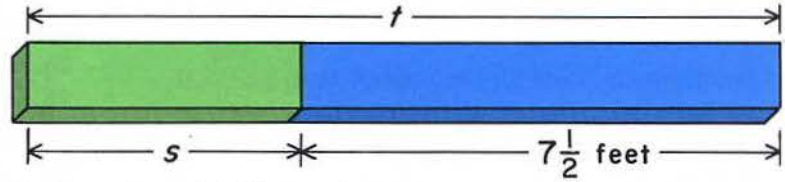
	a.	b.	c.	d.	e.	f.	g.
and the height of the dwarf in feet is	3	2	1	$2\frac{1}{2}$	$1\frac{1}{3}$	$\frac{7}{8}$	$\frac{11}{12}$
then the height of the giant in feet is	13	14	15	$13\frac{1}{2}$	$14\frac{2}{3}$	$15\frac{1}{8}$	$15\frac{1}{12}$

In the shorthand of algebra, we can write:

(a)	(b)	(c)	(d)
If ..... $d + g = 16$	If ..... $d + g = 16$	If ..... $d + g = 16$	$d + g = 16$
and if ..... $d = 3$	and ..... $d = 2$	and ..... $d = 1$	$d = 2\frac{1}{2}$
then ..... $g = 13$.	then ..... $g = 14$.	then ..... $g = 15$.	$g = 13\frac{1}{2}$

II. Bill has a board that is $7\frac{1}{2}$ feet long. He also has a shorter board. How far would the two boards reach if they are placed end to end? How long is the short board?

Again, we haven't enough information but we can fill out the chart that lists some of the possibilities.

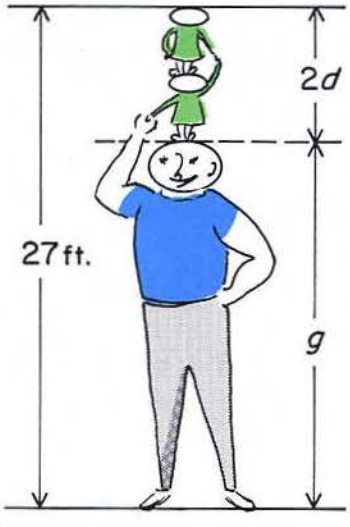


If the long board is $7\frac{1}{2}$ feet long

	a.	b.	c.	d.	e.
and the length in feet when placed end-to-end is	10	12	$8\frac{1}{2}$	14	$11\frac{1}{8}$
then the length in feet of the short board is	$2\frac{1}{2}$	$4\frac{1}{2}$	1	$6\frac{1}{2}$	$3\frac{5}{8}$

In the shorthand of algebra, we can write:

(a)	(b)	(c)	(d)
If ..... $s+7\frac{1}{2} = t$	If ..... $s+7\frac{1}{2} = t$	If ..... $s+7\frac{1}{2} = t$	$s+7\frac{1}{2} = t$
and ..... $s = 2\frac{1}{2}$	and ..... $t = 12$	and ..... $s = 1$	$t = 14$
then ..... $t = 10$.	then ..... $s = 4\frac{1}{2}$.	then ..... $t = 8\frac{1}{2}$.	$s = 6\frac{1}{2}$



III. Two dwarfs of the same size stand, one on top of the other, on the head of a giant. The top of the head of the higher dwarf is 27 feet from the ground. Without more information, we cannot know the height of the dwarfs or the giant. But —

If a giant and 2 dwarfs together are 27 feet tall

	a.	b.	c.	d.	e.
and if the giant's height in feet is	20	24	$21\frac{1}{2}$	$22\frac{1}{3}$	$25\frac{1}{4}$
then each dwarf's height in feet is	$3\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{3}{4}$	$2\frac{1}{3}$	$\frac{7}{8}$

(a)	(b)	(c)
If ..... $g + 2d = 27$	$g + 2d = 27$	$g + 2d = 27$
and ..... $g = 20$	$g = 24$	$g = 21\frac{1}{2}$
then ..... $2d = 7$	$2d = 3$	$2d = 5\frac{1}{2}$
and ..... $d = 3\frac{1}{2}$	$d = 1\frac{1}{2}$	$d = 2\frac{3}{4}$

Notes on Pupil Page 114

Tasks: *Pupils complete tables and then solve problems using the shorthand of algebra.*

Purpose: To help pupils become familiar with algebraic shorthand.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: This pupil page deals with very simple ideas. It is designed to help pupils become familiar with the use of algebraic shorthand. At first, the argument is presented in words, with accompanying pictures that help to illustrate the main idea. "A giant and a dwarf together are 16 feet tall. If the dwarf is 3 feet tall, how tall is the giant?" Very few pupils will have difficulty with this problem. The giant is 13 feet tall. Is it possible to check this answer? Yes, by adding the heights: 3 feet + 13 feet = 16 feet.

Essentially, the problem asks pupils to find two numbers that add up to 16. One of these numbers is the number of feet in the dwarf's height. Let this number be d. The other number is the number of feet in the giant's height. Let this number be g. This idea that the two numbers add up to 16 may be expressed by the equation $d+g=16$.

Full statement of I(a)

A giant and a dwarf together are 16 feet tall.
The dwarf is 3 feet tall.
So, the giant must be 13 feet tall.

Shorthand for I(a)

$$d+g=16$$
$$d=3$$
$$\therefore g=13$$

The sign $\therefore$ for *therefore* is a useful one for the pupils to know. It can also be read as "so it must be that."

Similarly, in exercise II we have:

Full statement of II(a)

A short board and a board $7\frac{1}{2}$ feet long are placed end to end.
The short board is $2\frac{1}{2}$ feet long.
So, the total length of the boards is 10 feet.

Shorthand for II(a)

$$s+7\tfrac{1}{2}=t$$
$$s=2\tfrac{1}{2}$$
$$\therefore t=10$$

Full statement of II(b)

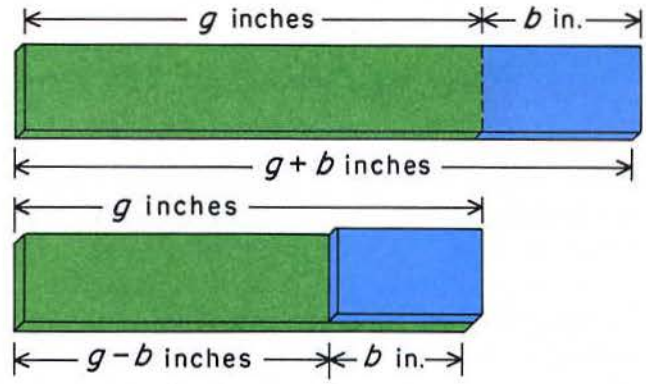
A short board and a board $7\frac{1}{2}$ feet long are placed end to end.
The total length of the boards is 12 feet.
So, the short board is $4\frac{1}{2}$ feet long.

Shorthand for II(b)

$$s+\tfrac{1}{2}=t$$
$$t=12$$
$$\therefore s=4\tfrac{1}{2}$$

I. Mary folded a long strip of paper and colored the region on one side of the fold green — on the other side, blue. She measured the length of the strip before it was folded — length of green plus length of blue. After it was folded, she measured the green part that was not covered by the blue.

She did this with many such strips and kept a record. Can you find the lengths of the blue and green regions in each example?



(All lengths are in inches.)

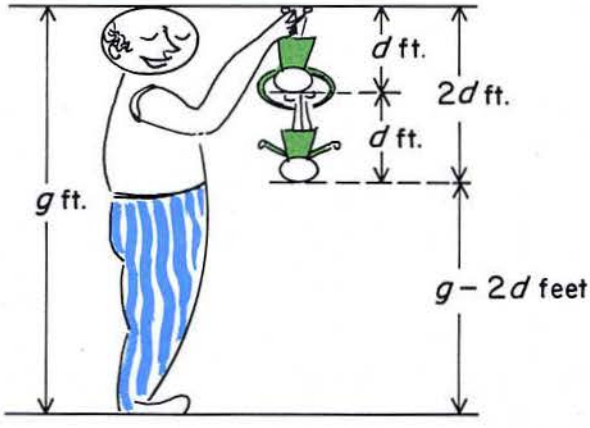
		a.	b.	c.	d.	e.	f.	g.	h.
Length of green and blue together	$(g+b)$	13	20	4	$15\frac{1}{2}$	$9\frac{1}{4}$	25	7	30
Length of green not covered by blue	$(g-b)$	7	10	2	$4\frac{1}{2}$	$4\frac{3}{4}$	11	4	12
Length of green	(g)	10	15	3	10	7	18	$5\frac{1}{2}$	21
Length of blue	(b)	3	5	1	$5\frac{1}{2}$	$2\frac{1}{4}$	7	$1\frac{1}{2}$	9

(a)
If.... $g + b = 13$
and.. $g - b = 7$
then..... $g =$ 10
and..... $b =$ 3 .

(b)
If.... $g + b = 20$
and.. $g - b = 10$
then..... $g =$ 15
and..... $b =$ 5 .

(f)
If..... $g = 18$
and..... $b = 7$
then.. $g + b =$ 25
and.. $g - b =$ 11 .

(e)
If..... $g = 21$
and.. $g - b = 12$
then..... $b =$ 9
and.. $g + b =$ 30 .



II. A giant touches the ceiling with his head. He holds a dwarf's feet against the ceiling. This dwarf holds the feet of another dwarf.

The chart below indicates several possibilities. Of course, giants and dwarfs can be any height we wish to imagine. (All heights are in feet.)

		A.	B.	C.	D.	E.	F.	G.	H.
Height in feet of giant	(g)	20	100	18	24	$33\frac{3}{4}$	47	$50\frac{1}{2}$	75
Height in feet of two dwarfs	$(2d)$	4	3	1	2	$3\frac{3}{4}$	9	6	$\frac{1}{2}$
Height of one dwarf	(d)	2	$1\frac{1}{2}$	$\frac{1}{2}$	1	$1\frac{7}{8}$	$4\frac{1}{2}$	3	$\frac{1}{4}$
Number of feet from bottom of dwarf's head to floor	$(g-2d)$	16	97	17	22	30	38	$44\frac{1}{2}$	$74\frac{1}{2}$

(B)
If..... $g = 100$
and..... $2d = 3$
then..... $d =$ $1\frac{1}{2}$
and.. $g - 2d =$ 97 .

(D)
If..... $d = 1$
and.. $g - 2d = 22$
then.... $2d =$ 2
and..... $g =$ 24 .

(F)
If..... $2d = 9$
and.. $g - 2d = 38$
then..... $g =$ 47
and..... $d =$ $4\frac{1}{2}$.

(G)
If..... $g = 50\frac{1}{2}$
and.. $g - 2d = 44\frac{1}{2}$
then..... $2d =$ 6
and..... $d =$ 3 .

Notes on Pupil Page 115

Tasks: *Pupils complete tables and then solve problems using algebraic shorthand.*

Purpose: To help pupils become familiar with algebraic shorthand.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: It is possible to illustrate what we mean by the shorthand of algebra in the following way:

How we think the problem out

A giant and 2 darwfs together are 27 feet tall.
The giant takes up 20 feet.
So, 2 dwarfs must fill the other 7 feet.
So, each dwarf must be $3\frac{1}{2}$ feet high.

Shorthand form

$g+2d=27$
$g=20$
$\therefore 2d=7$
$\therefore d=3\frac{1}{2}$

It may help to write out full descriptions together with the algebraic shorthand.

Full statement of exercise I(b)

The full strip is 20 inches long.
The green region is 10 inches longer than the blue.
So, the green region is 15 inches long, and the blue region is 5 inches long.

Shorthand for exercise I (b)

$g+b=20$
$g-b=10$ (or $g=b+10$)
$\therefore g=15$
$\therefore b=5$

A useful way of solving this problem is to notice (from the drawing) that the $(g-b)$-inch strip is $2b$ inches shorter than the $(g+b)$-inch strip. So, $2b=20-10=10$ and $b=5$. It is also possible to notice that the $(g+b)$-inch strip together with the $(g-b)$-inch strip is twice as long as the g-inch strip. So, $20+10=2g$ and $g=15$.

Pupil page 115 involves ideas expressed on page 114. The problems are to be solved by common sense. By filling in the spaces on this page, the pupil gains practice in the use of mathematical shorthand.

Headline

I.

$$a + b = 12$$
$$a - b = 2$$
$$a = 7$$
$$b = 5$$

Example: Mrs. Clark made two purchases. Her first purchase ($\$a$) was \$2 more than the second ($\$b$). Altogether, she spent \$12 ($a + b$ dollars). Therefore, the amount of her first purchase was \$ 7 ; the second was \$ 5 .

Make up another example to fit the headline: _____

II.

$$x + 2y = 130$$
$$y = 20$$
$$2y = 40$$
$$x = 90$$

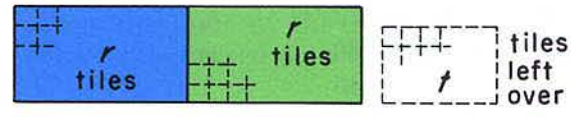
Example: Home — x mi. — Grandmother's — y mi. — Uncle Ed's; $2y$ mi. back to Grandmother's.

After leaving home, we drove to Grandmother's; then to Uncle Ed's; then back to Grandmother's. Altogether, we drove 130 miles. It is 20 miles from Grandmother's to Uncle Ed's. From Grandmother's, we shall have to drive 90 miles to get home.

Make up another example to fit the headline: _____

III.

$$2r + t = 144$$
$$t = 24$$
$$2r = 120$$
$$r = 60$$



Mr. Brown bought 144 square tiles. He had two floors the same size to cover. When he finished, there were 2 dozen tiles left over. He used 60 tiles for each floor.

Make up another example to fit the headline: _____

IV.

$$x - y = 60$$
$$x + 2y = 75$$
$$x = 65$$
$$2y = 10$$
$$y = 5$$

Jack and Bill each weighed the same when school started in the fall. Jack was sick for several weeks and lost some weight, and by Thanksgiving he weighed only 60 pounds. Bill gained twice as much as Jack lost, and he weighed 75 pounds. When school started, each must have weighed 65 pounds. Jack lost 5 pounds, and Bill gained 10 pounds.

Make up another example to fit the headline: _____

Notes on Pupil Page 116

Tasks: *Pupils make up problems suggested by the algebraic headlines, patterned after the illustrative problems.*

Purpose: To test pupil understanding of the relationship of algebraic sentences to the real world.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

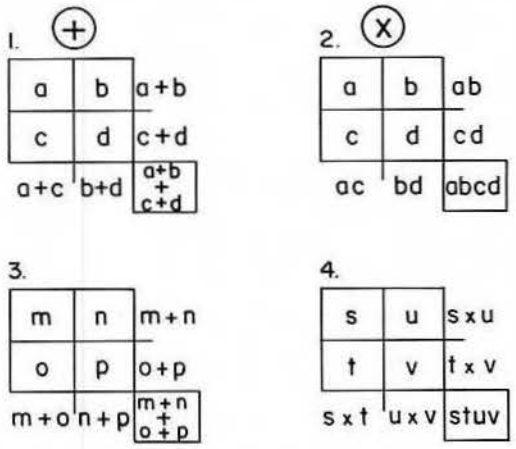
The Lesson: On this page, pupils are asked to make up a problem that leads to the given shorthand. This approach is used to test the understanding pupils have of the basic idea. In the first problem the shorthand notations $a+b$ and $a-b$ are given. Any problem that illustrates addition and subtraction for the same pair of numbers would be appropriate. For example:

Heights: Art and Bill together (like a giant and dwarf) are 12 feet tall. Art is 2 feet taller than Bill.

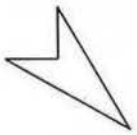
Money: Anne and Betty have 12 dollars. Anne has 2 dollars more than Betty.

Speed: A motor boat travels at 12 knots with the current, but at only 2 knots against the current. (So its speed in still water is 7 knots, and the current is 5 knots.)

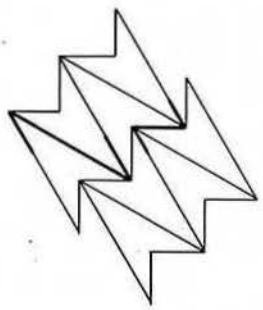
Here is another use for algebraic shorthand. Have pupils complete *Cross Number Puzzles* like the following:



Here is a geometric diversion. Ask pupils, "Could you cover a floor (that is, a plane) if your tiles were this shape?"



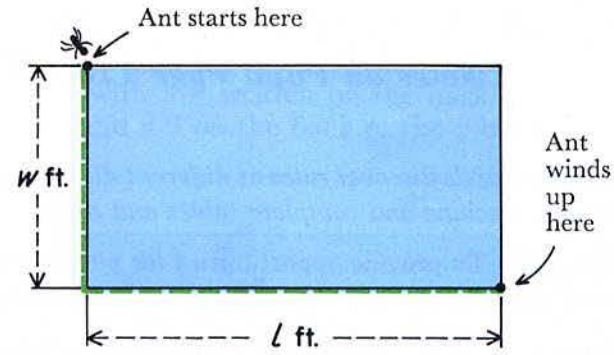
The answer is, "Yes." You can use exactly the same procedure as on page 55. Here is the pattern for "fitting" the tiles:



Quite a sophisticated pattern. Note examples of the Z-result in the pattern.

I. An ant started crawling from one corner of a rectangularly shaped room. He crawled along the wall the width (w feet) of the room plus the length (l feet) of the room. It took exactly 360 1-foot square tiles to cover the floor. How far did the ant crawl?

We do not have enough information, but we can work out several of the possibilities.



The area of floor is 360 square feet, or $l \cdot w = 360$.

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
The length of the room.	30	24	60	20	36	120	72	90	40	45
The width of the room.	12	15	6	18	10	3	5	4	9	8
The distance the ant crawled.	42	39	66	38	46	123	77	94	49	53

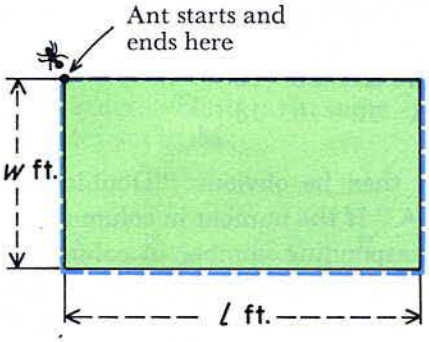
In the shorthand of algebra, we can write:

(1)
If.... $l \cdot w = 360$
and..... $l = 30$
then..... $w =$ 12
and.. $l + w =$ 42.

(4)
If.... $l \cdot w = 360$
and.. $l + w = 38$
then..... $l =$ 20
and..... $w =$ 18

(6)
If..... $l \cdot w = 360$
and..... $w = 3$
then..... $l =$ 120
and.. $l + w =$ 123.

(9)
If... $l \cdot w = 360$
and.. $l + w = 49$
then..... $l =$ 40
and..... $w =$ 9



II. Now suppose that the ant crawls all the way around the room — back to the corner from which he started. All we know about the floor of the room is that it is rectangular in shape and covered with whole tiles that are 1-foot square.

The chart below considers some of the possibilities.

	1.	2.	3.	4.	5.	6.	7.	8.
The length of the room in feet (l).	10	12	20	15	11	9	17	13
The width of the room in feet (w).	15	9	17	15	7	8	13	11
The area of the room in square feet ($l \cdot w$).	150	108	340	225	77	72	221	143
The distance the ant crawls from one corner to opposite corner ($l + w$).	25	21	37	30	18	17	30	24
Twice the distance the ant crawls $2(l + w)$.	50	42	74	60	36	34	60	48
The total distance the ant crawls $(2l + 2w)$.	50	42	74	60	36	34	60	48

III. Suppose that the ant crawled around three of the walls and had 11 feet to go before reaching his starting place. The trip around the three walls had been 25 feet. The room must be 7 feet wide.

In the shorthand of algebra, we could write:

$w + l + w = 25$
$2w + l =$ 25
$l =$ 11
$2w =$ 14
$w =$ 7

Tasks: *Pupils complete tables.*

Purpose: To help pupils become familiar with algebraic shorthand as a compact representation of a story problem. To focus attention on the notions of perimeter and area.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations; geometry; measurement.

The Lesson: It may be necessary to remind students that $l \cdot w$ (or lw) means l times w. In the example where $l=30$ and $w=12$, it is understood that the room contains 30 rows of 12 tiles, or 30 times 12 in all. The general rule is that the area of the floor (in tiles) is l times w: the number of tiles in the length of the room multiplied by the number in the width.

Pupil page 117 requires some systematic guessing. In the first table, the fourth column shows that the ant crawled 38 feet. So, we wish l and w to add up to 38. Since there are 360 tiles in the room, l times w must be 360. We may not find it easy to guess the numbers. It pays to make a complete list of the possibilities. 360 tiles could be 1 row of 360 tiles, or 2 rows of 180, or 3 rows of 120, and so on. We can collect these data in the table above.

w	l	$l+w$
1 row of	360 tiles	Ant crawls 361 feet
2 rows of	180 tiles	Ant crawls 182 feet
3	120	123
4	90	94
5	72	77
6	60	66
8	45	53
9	40	49
10	36	46
12	30	42
15	24	39
18	20	38

In problems II and III the area of the floor (in tiles) is not specified. In the first column, pupils are given both l and w. In the second column, w is 9 and $l+w$ is 21, and so l must be 12—the number which, added to 9, gives 21.

In the third column, there are 340 tiles and the room is 20 tiles long. So there must be 17 rows and the room must be 17 feet wide.

The fourth column calls for some guessing or making of tables. The distance around the room is 60 feet. From one corner to the opposite (around the edge) is just half this, or 30 feet. So we write 30 in the row for $l+w$. Now we are looking for numbers for l and w whose sum is 30 and whose product is 225. The only pair is 15 and 15.

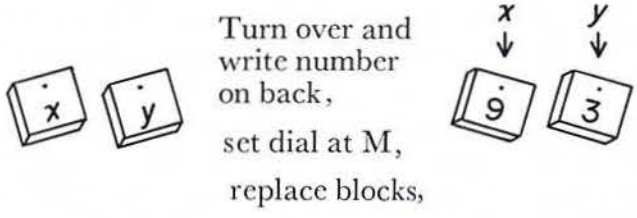
In the fifth column, there are two solutions. The room could be 77 feet by 1 foot, or 11 feet by 7 feet. Logically, we must admit both solutions as correct. In practice, a room 1 foot wide is unlikely, and the only reasonable solution is that $l=11$ and $w=7$.

In the sixth column, since the perimeter is 34 feet, it follows that $l+w$ is 17. So, since $w=8$, $l=9$.

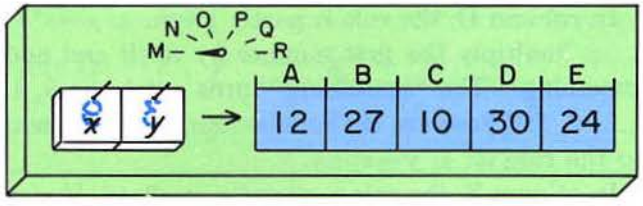
The seventh and eighth columns resemble the fifth. Logically, we have to admit that the room in the seventh column could be either 17 feet by 13 feet or 221 feet by 1 foot and that the room in the eighth column could be either 13 feet by 11 feet or 143 feet by 1 foot.

Another Machine left by Alec Marson for the Science Fair

I. This machine has two removable blocks marked x and y. To operate it, you remove the blocks and write a number on the back of each; set the dial or pointer; replace the blocks — and numbers will appear immediately on each of the five blue screens.



and the machine looks like this:



Keep changing the numbers written on the blocks and keep a record of the numbers the machine writes. Perhaps you can discover the rules. Here are some results with the pointer set at M:

x	y		A	B	C	D	E
9	3	→	12	27	10	30	24
4	6	→	10	24	8	27	20
11	2	→	13	22	11	25	26
1	5	→	6	5	4	8	12

Complete the chart above and the rules below:

Rule A: $x, y \rightarrow x + y$

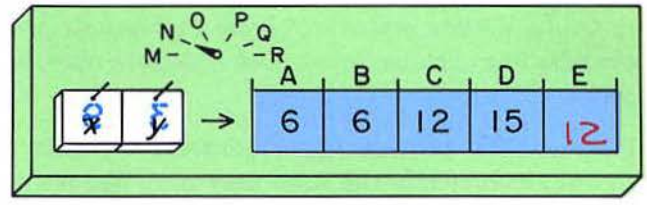
Rule B: $x, y \rightarrow x \cdot y$ OR xy

Rule C: $x, y \rightarrow x + y - 2$

Rule D: $x, y \rightarrow xy + 3$

Rule E: $x, y \rightarrow 2(x+y)$ OR $2x + 2y$

II. When the pointer was moved into position N, the following results were noted. Tests began with a 9 written on the back of the x-block and a 3 on the back of the y-block.



x	y		A	B	C	D	E
5	4	→	1	8	2	13	2
13	7	→	6	14	12	27	12
13	9	→	4	18	8	31	8
10	3	→	7	6	14	16	14

Rule A: $x, y \rightarrow x - y$

Rule B: $x, y \rightarrow 2y$

Rule C: $x, y \rightarrow 2x - 2y$

Rule D: $x, y \rightarrow x + 2y$

Rule E: $x, y \rightarrow 2(x - y)$

III. Here are results with the pointer set at O:

x	y		A	B	C	D	E
10	5	→	25	25	15	25	15
4	8	→	22	16	24	28	0
7	3	→	20	17	9	16	11
11	0	→	21	22	0	11	22
6	9	→	25	21	27	33	3

Rule A: $x, y \rightarrow x + y + 10$

Rule B: $x, y \rightarrow 2x + y$

Rule C: $x, y \rightarrow 3y$

Rule D: $x, y \rightarrow x + 3y$

Rule E: $x, y \rightarrow 2x - y$

Notes on Pupil Page 118

Tasks: *Pupils discover rules at different dial settings on the machine and complete tables and rules.*

Purpose: To provide opportunities for pupils to record algebraically the rules they have discovered.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Rule A is given near the bottom of the page: $x, y \rightarrow x+y$; that is, "add the two numbers." This agrees with $9+3 \rightarrow 12$. Under the 12, we fill in a 10, since 10 is $4+6$. We have to wait before we can fill in any more of column A.

In column B, the examples $9, 3 \rightarrow 27$ and $4, 6 \rightarrow 24$ suggest multiplication. Rule B in shorthand is $x, y \rightarrow x \cdot y$. The third entry in Column B shows that the missing number is 2, since 2 is the only number whose product with 11 is 22. Since $11+2=13$, it is possible to enter 13 in column A, third row.

In the fourth row, the number 6 appears in column A, and the number 5 appears in column B. Therefore, 6 is the sum of the missing numbers and 5 is their product. There is only one way to factor 5 using whole numbers: 5×1. And since $5+1=6$, 5 and 1 must be the missing numbers.

Rule C is given near the bottom of the page. $x, y \rightarrow x+y-2$; that is, add the numbers and subtract 2. The quickest way to do this is to take 2 from each number in column A. This also provides a clue which may be needed in order to discover the rule for column D. Since column C has something to do with column A, perhaps column D has something to do with one of the other columns. Yes, it has. The numbers 30, 27, 25 are 3 more than 27, 24, 22 in column B. So we put an 8 in the empty space. The rule for D is: "add 3 to the number in column B." But the number in column B has already been given in shorthand as $x \cdot y$; so rule D can be stated as "add 3 to $x \cdot y$." In shorthand this would be written: $x \cdot y+3$ or $3+x \cdot y$.

What about column E? If pupils' minds have become fixed on addition and subtraction, they may start comparing column E with the other columns in the following ways: Are the numbers in column E 6 less than those in D? No. Perhaps they are 3 less than those in B? No. How about 12 more than those in A? No. Pupils will eventually discover that the numbers in column E are not found by adding a fixed number to or subtracting a fixed number from each of the numbers in another column. This conclusion will force pupils to look for another approach, and so ask themselves: what is there in arithmetic besides adding and subtracting? Or, comparing column E with each other column, they might observe that, in one case, there is a definite relationship:

E	24	20	26
A	12	10	13

The rule should then be obvious: "Double the number in column A." If the number in column A is $x+y$ then the corresponding number in column E is $2(x+y)$. The rule for column E is $x, y \rightarrow 2(x+y)$.

IV. Here are results with the pointer at P:

x	y		A	B	C	D	E
7	3	→	49	9	40	52	18
5	4	→	25	16	9	29	32
8	1	→	64	1	63	65	2
10	6	→	100	36	64	106	72
9	5	→	81	25	56	86	50

Rule A: $x, y \rightarrow x \cdot x$ or x^2

Rule B: $x, y \rightarrow y^2$

Rule C: $x, y \rightarrow x^2 - y^2$

Rule D: $x, y \rightarrow x^2 + y$

Rule E: $x, y \rightarrow 2 \cdot y^2$ or $2y^2$

V. In the blue-tinted area are some expressions we have been using to complete the rules on these pages.

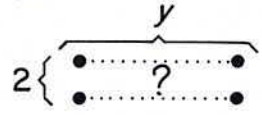
x^2 $\quad x + 3y \quad$ $x^2 - y^2$

$2x + 2y \quad y^2 \quad x + y - 2$

$2y \quad 2(x + y)$

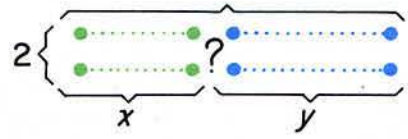
Find sketches below that suggest these ideas.

(A)



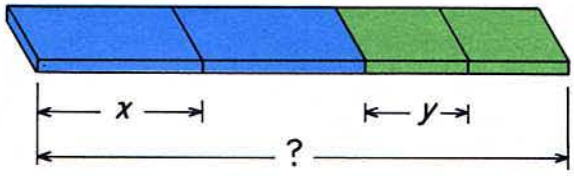
This sketch suggests the idea of 2y.

(B)



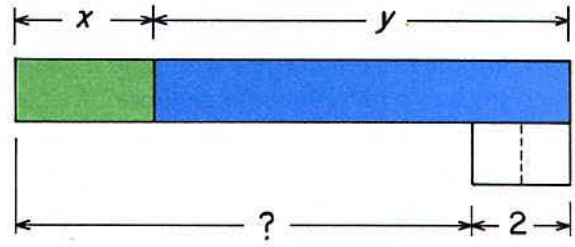
This suggests the idea of 2(x+y).

(C)



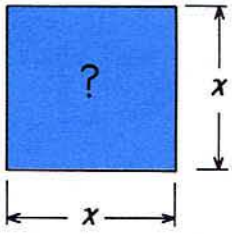
This suggests the idea of 2x+2y.

(D)



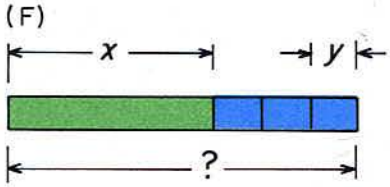
The sketch above suggests the idea of x+y=2.

(E)



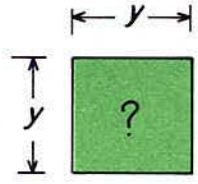
This sketch suggests the idea of x·x OR x^2.

(F)



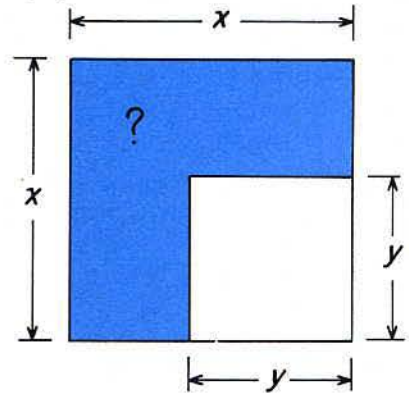
The above sketch suggests the idea of x+3y.

(G)



The sketch above suggests the idea y^2.

(H)



The blue part of the sketch above suggests the idea of $x^2 - y^2$.

Notes on Pupil Page 119

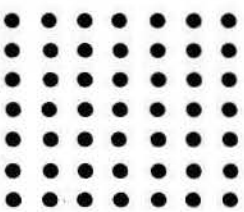
Tasks: *In IV pupils discover rules at different dial settings on the machine and complete tables and rules. In V they match algebraic expressions from the blue box with pictures which illustrate these expressions.*

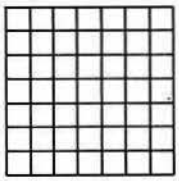
Purpose: To provide opportunities for pupils to record algebraically the rules they have discovered. To test pupil understanding of the relationship of algebraic expressions to physical representations.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: When the pointer is set at P, the rules are based on the idea of multiplying a number by itself: 3 times 3, 5 times 5, or x times x. In mathematical terms this idea would be expressed by: $3 \cdot 3$, $5 \cdot 5$, and $x \cdot x$, or by: 3^2, 5^2, and x^2.

On pupil page 119, pupils are told the rule for column A: $x, y \rightarrow x \cdot x$; that is, the number in the x column is to be multiplied by itself. In the first row, the number is 7. So 49 should be written in column A, since 49 is 7 times 7, or 7^2. You may wish to remind students that 7^2 is read "seven square(d)," because it can be demonstrated by use of squares:





In column B, 9 is $3 \cdot 3$, or 3^2; 16 is $4 \cdot 4$, or 4^2; and 1 is $1 \cdot 1$, or 1^2. So, $x, y, \rightarrow y \cdot y$ (or y^2) is the shorthand for rule B.

In column C, the number in column B is subtracted from the number in column A. If x^2 is the number in column A and y^2 is the number in column B then $x, y \rightarrow x^2 - y^2$ is the rule for column C. In the first row, for example, the numbers to be considered are 7 and 3 and the procedure is: $7 \times 7 - 3 \times 3 = 49 - 9 = 40$.

In column D, the rule is partly given: $x, y \rightarrow x^2 +$ ___; "multiply the first number by itself and add something." The "something" turns out to be 3, 4, ___, 6, 5. These are the numbers in the y column. So the rule is: $x, y \rightarrow x^2 + y$.

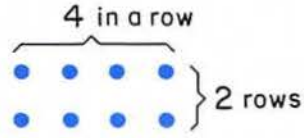
In column E, the rule is given: $x, y \rightarrow 2 \cdot y^2$. If y^2 *is* the number in column B, the corresponding number in E is $2 \cdot y^2$.

Pupils are liable to misinterpret $2y^2$ as: double y and square the result. For example, if y is 10, they would double 10 and square the result, getting 400. But this is not what is meant by 2×10^2. The conventional interpretation of $2y^2$ is: square y and double the result. Thus, to find 2×10^2 we square 10 and double the result: $2 \times 100 = 200$.

The rest of the page tests the pupil's understanding of the algebra he has used by asking him which picture goes with which shorthand rule.

We can use sketches to illustrate many of the ideas suggested by the shorthand of algebra.

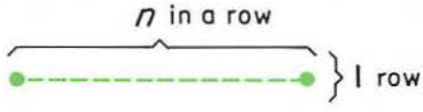
In arithmetic, we use sketches or diagrams to see clearly what is going on. For example: there are four trees in each of two rows. A sketch might be:



Because $2 \times 4 = 8$, there would be eight trees in all.

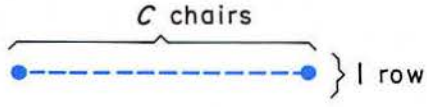
In some situations we don't have enough information to draw such a complete sketch. However, we can indicate at least as much as we do know. Here is an example:

Think of a number. Arrange that many counters in a single row. Only you know what number you thought of. Others can use a letter such as n to take the place of that number until they find out for sure what it is. Here is a sketch of the idea:

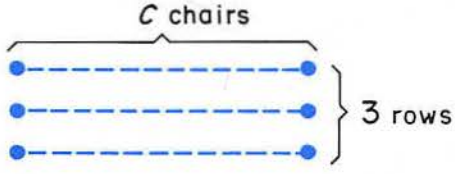


This sketch will serve well unless the number you thought of was 0.

Think once again of a number. Arrange that many chairs in a row. We can say that you have arranged c chairs in a row, and sketch it as:

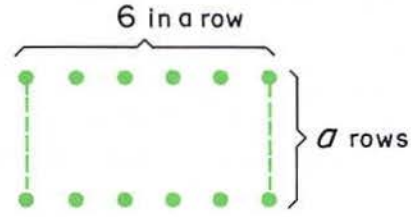


Next, arrange that same number of chairs in each of three rows:



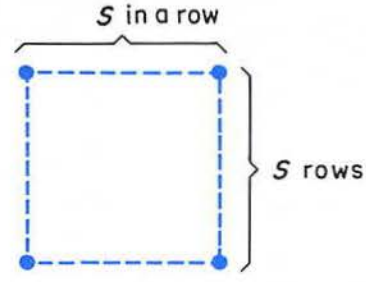
We can say that you arranged c chairs in each of 3 rows. So, there are $3 \times c$ chairs or $3 \cdot c$ or $3c$ chairs.

Think of another number. Draw that number of rows with 6 dots in each row. We shall use an a to remind us of the number you thought of.



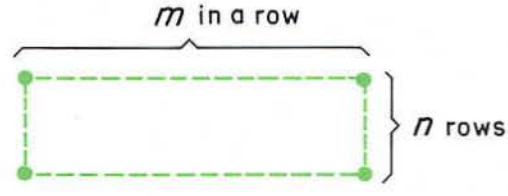
You must have drawn $6 \times a$ or $6a$ dots.

Think of a number. (We shall use an s to remind us of that number until we know more about it.) Put that number of counters in a row. Add rows until you have that same number of rows.



You must have, in all, $s \times s$ counters or s^2 counters; "s^2" is read as "s squared."

Think of a number. (We'll think of it as m.) And, once again, think of a number. (We'll use n to remind us of it.) Now draw n rows of dots with m dots in each row.



This sketch reminds us that you must have drawn $m \times n$ or mn dots.

Notes on Pupil Page 120

Tasks: *Pupils read and study the explanations.*

Purpose: To help pupils understand how to illustrate the ideas suggested by the shorthand of algebra.

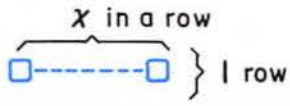
Unifying Ideas: Structure; number and counting; numeration; multiplication and division; functions and relations; geometry.

The Lesson: Many pupils will read this page without any difficulty. The purpose of the pictures is to convey the meaning of expressions such as $x+y$, $5a$, mn, and s^2. If a pupil has difficulty understanding the use of letters, it may help to have him begin with familiar results in arithmetic. For example: Five rows with 3 dots in each illustrates 5×3; what would 5 rows with 7 dots in each illustrate? (The answer 35 is less appropriate. The picture is intended to portray a physical interpretation of multiplication.) How would you illustrate 5×8? By 5 rows with 8 dots in each. How would you illustrate 5×13? By 5 rows of 13 dots. Could we do this with 5 times any whole number? Yes. Alf is going to choose a number. How would you illustrate 5 times the number chosen by Alf? By 5 rows, each containing the number of dots that Alf has selected. What would be a good shorthand to show a place for Alf's number? We could use a. How would we write 5 times Alf's number? As $5 \cdot a$ or $5a$. What could we use to picture $5a$? Rows of dots. How many rows? Five. How many dots in each row? Whatever number Alf chose. So the rectangle has 5 rows with a dots in each row.

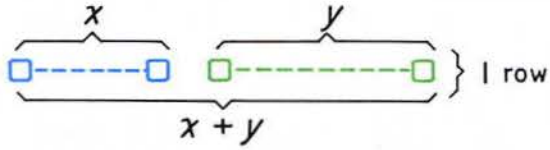
When pupils are ready for it, another step can be taken. We have used 5 in all our questions above. Is there something special about 5? No. We could have chosen any other number and drawn that number of rows of dots. Let's have Betty choose the number of rows that there are going to be. We use b to hold a place for the number Betty chooses. Our rectangle now has b rows and a dots in each row. It gives us a picture of $b \cdot a$ or ba.

When we explain to pupils that an array of 5 rows of 3 dots helps us to think of 5×3, the pupils soon realize that 5 and 3 were chosen at random. We could use an array to illustrate the product of any two (whole) numbers. Indeed, this is an important point in the teaching of arithmetic. Since, as adults, pupils may never actually have to use the numbers used in some particular problem they did in school, we try to make it clear to them that a method employed with particular numbers can be applied to any numbers. Even in Level A, we have tried to convey general ideas about arithmetic. This means that from the beginning we have been concerned with algebra. All we are trying to do now is to give students a convenient way of writing what they already know.

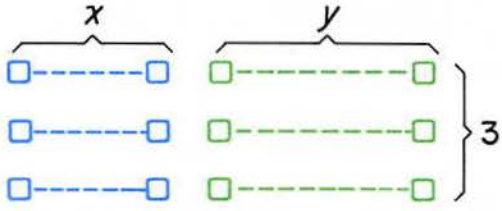
Think of a number. Draw that many blue squares in a row.



Think of another number. Extend the row with that number of green squares.

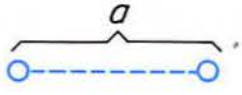


There are $x + y$ squares in a row. Now, draw 3 such rows.

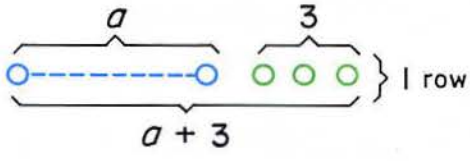


The sketch tells us that you must have drawn $3x + 3y$ squares. Or, we can say there must be $3(x + y)$ squares.

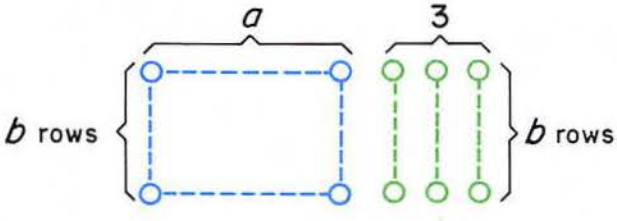
Think of a number and draw a row with that number of circles in it.



Add 3 circles to that row.



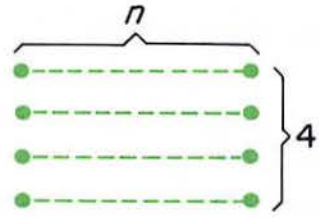
You must have drawn $a + 3$ circles. Now think of a number (b) and draw that number of rows like the one above.



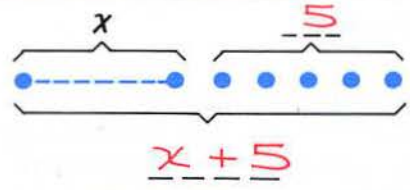
There are $a \times b$ or ab blue circles. There are $3 \times b$ or $3b$ green circles. There are $ab + 3b$ circles altogether.

Or, we can say that altogether there are $b(a + 3)$ circles.

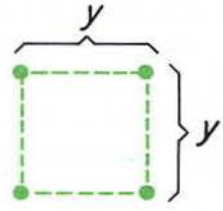
Each diagram below helps indicate a certain idea. Complete the sketch and the shorthand that suggests the same idea.



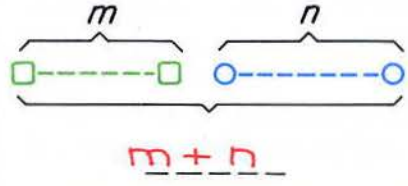
4 x n or 4n



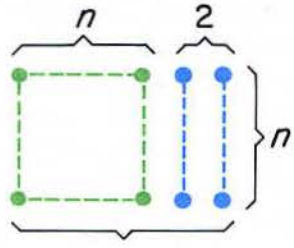
x + 5



y · y or y^2



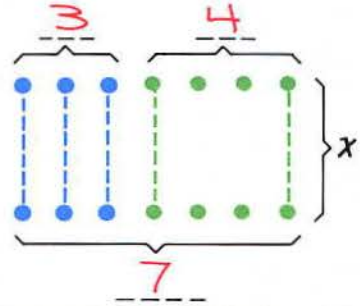
m + n



$n^2 + 2n$

or

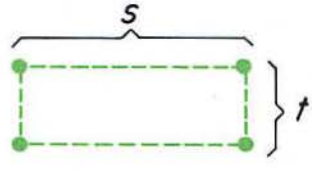
$n(n+2)$



$3x + 4x$

$x(3+4)$

or $7x$



s · t OR st

Notes on Pupil Page 121

Tasks: *Pupils read and study the explanations. Pupils then complete the sketches and shorthand illustrated by the sketches.*

Purpose: To help pupils understand how to illustrate the ideas suggested by the shorthand of algebra.

Unifying Ideas: Structure; number and counting; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: On this page, pictures are used to demonstrate some results in algebra. The third sketch represents $3 \cdot (x+y)$ or $3x+3y$. The parentheses are needed to indicate 3 times the sum of x and y; without parentheses, the expression would indicate the sum of $3x$ and y. But since there are 3 groups of x blue objects and 3 groups of y green objects, the number of objects can be indicated by the expression $3x+3y$.

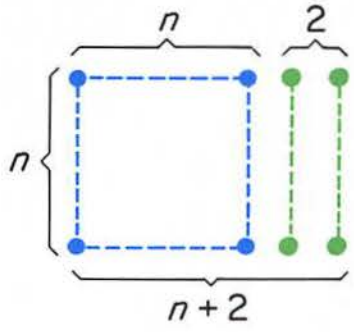
Next, a picture is used to find another way of writing $b\ (a+3)$. Note here the question of order in multiplication. Given an array with b rows of a objects, pupils may write $b \cdot a$ or ba to designate b times a. Or they might describe it as a columns with b objects in each, and write $a \cdot b$ or ab for a times b. Either form, ba or ab, is correct for the product. Usually, because of alphabetical order, ab is preferred to ba. But do not make an issue of this. Mathematically, ab and ba mean exactly the same thing.

The first four pictures in the right-hand column involve nothing new. In the fifth picture there is a square array containing n^2 objects, and two columns, each containing n objects. This could be described as representing n^2+2n objects. It is also correct to say that there are n rows with $n+2$ objects in each, so there are $n(n+2)$ objects in all. It follows that n^2+2n and $n(n+2)$ are equivalent.

The sixth picture can be described in several ways. It can be described as 3 columns with x objects in each and 4 columns with x objects in each, making $3x+4x$ objects in all. This can also be described as x rows with $3+4$ objects in each, making $x(3+4)$ objects. Or, it can be described as 7 columns with x objects in each, making $7x$ objects; as x rows with 7 objects in each, making $x7$ objects. Each of these descriptions is correct.

Jane said to Al, Jack, Betty, Dick, and Ruth, "Each of you think of a number. Draw a row with that number of dots. Now add 2 more dots to the row.

"Think again of the number you first thought of and draw that many rows just like your first row. I don't know what you've drawn, but a sketch helps me see clearly what I do know.



"I use an n to remind me of whatever number you thought of. So, I can say that you have drawn $n^2 + 2n$ dots. Or, I can say that you have drawn $n(n + 2)$ dots.

"Each of you fill in this chart and see if I am not right.

	Al	Jack	Betty	Dick	Ruth
Your number n	5	8	11	7	20
n^2	25	64	121	49	400
$2n$	10	16	22	14	40
$n^2 + 2n$	35	80	143	63	440
n	5	8	11	7	20
$n + 2$	7	10	13	9	22
$n(n + 2)$	35	80	143	63	440

"No matter what whole number you think of," Jane explained, "my sketch remains the same. And it suggests that, for each whole number n,

$$n(n + 2) = n^2 + 2n$$

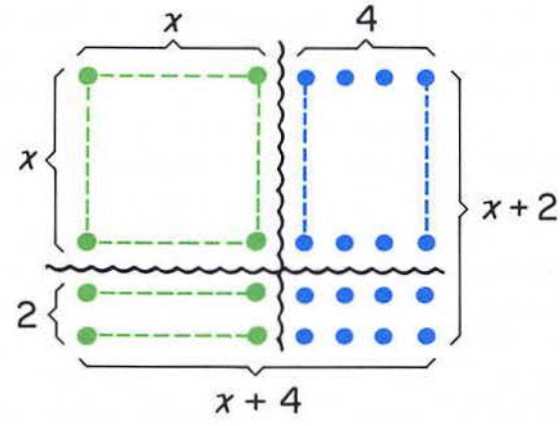
"Try it with any whole number you can think of. The results will be the same."

Jane asked her friends to try another experiment with her.

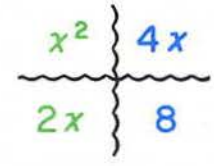
"Think of another whole number. I'm going to use an x to remind me of it. Draw that many dots in the top row. Then add 4 more.

"Next, draw as many rows like your first row as the number you first thought of.

"My sketch looks much like the earlier one. But, now add 2 more rows. There is my sketch:



"Another diagram leads me to the conclusion that, for each whole number x,



$$(x + 2)(x + 4) = x^2 + 6x + 8$$

"Test my conclusions in the chart below."

	Al	Jack	Betty	Dick
x	6	9	3	5
$x + 2$	8	11	5	7
$x + 4$	10	13	7	9
$(x+2)(x+4)$	80	143	35	63
x^2	36	81	9	25
$6x$	36	54	18	30
$x^2 + 6x + 8$	80	143	35	63

Make your own chart like the one above. Think of other whole numbers. Keep experimenting until you are completely confident that there will be no surprises.

Notes on Pupil Page 122

Tasks: *Pupils complete two charts and then make up charts of their own.*

Purpose: To teach pupils to interpret algebraic expressions illustrated by diagrams.

Unifying Ideas: Structure; number and counting; functions and relations; addition and subtraction; multiplication and division.

The Lesson: Jane's first picture suggests that, for any number n, $n^2+2n=n(n+2)$. Following her instructions, the other pupils are to choose particular numbers and draw pictures to illustrate them. Al would draw a 5×5 array and 2 columns with 5 dots in each; this would show that $5\times(5+2)=5\times5+5\times2$. In the table, each pupil works out the expressions n^2+2n and $n(n+2)$ for the number n he has chosen and finds that the two expressions are equivalent.

Jane's second sketch suggests that, for any number x, $x+2$ rows with $x+4$ dots in each can be broken up into the following parts: a square array x by x; six lines of x dots (four upright and two lying on their sides); and a rectangular array of 8 dots. This shows that, for any x, $(x+2)(x+4)=x^2+6x+8$.

The table indicates that Al chose 6. Jack had 11 for $x+2$; so he must have started with 9. Dick had 25 for x^2; so he must have started with 5. Betty's sketch would have 35 dots in it—either 35×1 dots or 5×7 dots. Since 35×1 is impossible (there have to be at least 2 rows of dots and 4 columns), it must be 5×7. And, if $x+2$ is 5 and $x+4$ is 7 then Betty must have chosen 3.

Here is a supplementary activity. Insight into mathematical sequences comes from formulating pattern sentences that indicate the nature of the sequences.

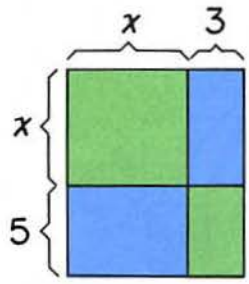
	n	1	2	3	4	5	6	7	8
sequences	a	1	3	5			11		
	b	0		6	9		15		
	c		6		16	21	26		
	d		8			32	40		
	e	98	96	94					

Pattern Sentences	Answers
a = 2n−	(2n−1)
b =	(3n−3)
c = 5n−	(5n−4)
d =	(8n−8)
e = 100−	(100−2n)

Once a pattern sentence has been determined it is, of course, then possible to determine the value of any term in the sequence without computing any of the preceding terms. Thus, the twentieth terms in the sequences above are:

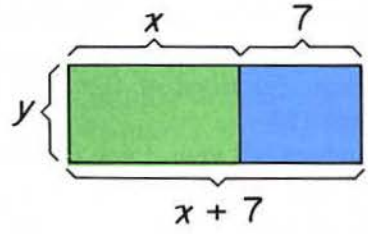
$2(20)-1=39$
$3(20)-3=57$
$5(20)-4=96$
$8(20)-8=152$
$100-2(20)=60$

Drawing dots gets tiresome. Another kind of sketch might look like this:



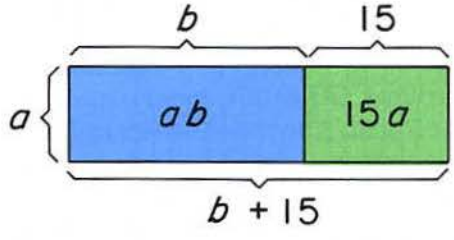
x^2	$3x$
$5x$	15

$(x + 3)(x + 5) = x^2 + \underline{8}\,x + \underline{15}$

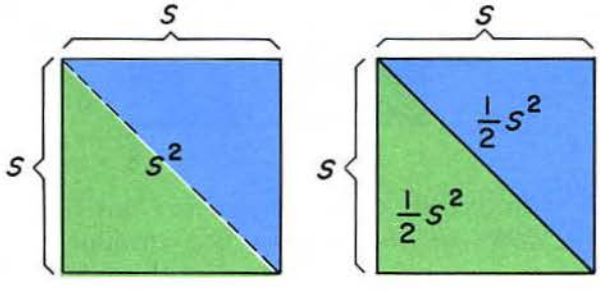


xy	$7y$

$xy + \underline{7y} = y(\underline{x} + \underline{7})$

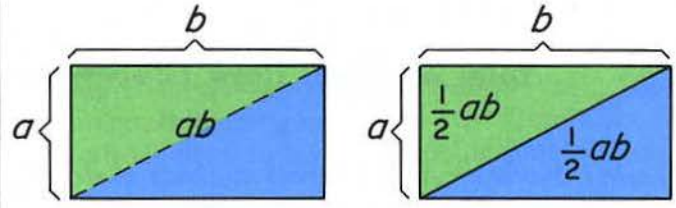


$ab + \underline{15a} = a(\underline{b} + \underline{15})$



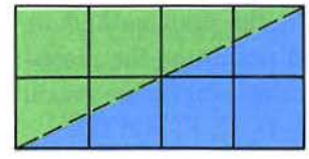
$\frac{1}{2}s^2 + \frac{1}{2}\underline{s^2} = \underline{s^2}$

The blue and green regions together suggest s^2. The green region suggests $\frac{1}{2}s^2$. The blue region also suggests $\frac{1}{2}s^2$.

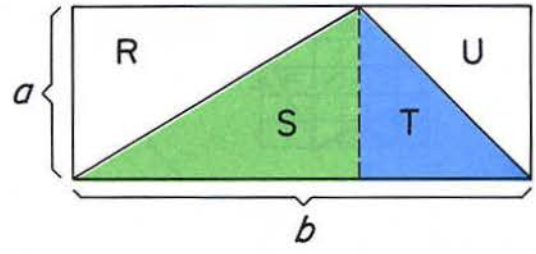


$\frac{1}{2}\underline{ab} + \frac{1}{2}\underline{ab} = \underline{ab}$

a and b remind us of two numbers. If we had known that those numbers were 2 and 4, we could have drawn the following sketch:



It would take $\frac{1}{2}$ of 2×4 small squares to cover the green region — 4 squares; and $\frac{1}{2}$ of 2×4 to cover the blue region. This is a special example of the idea of the sketch at the top of this column.



The large rectangle is a sketch that suggests ab.

Is the region labeled R the same size as the green region labeled S? YES

Is region U the same size as the blue region T? YES

Are regions R and U together the same size as regions S and T together? YES

Can we say that the colored region (blue and green together) suggests $\frac{1}{2}ab$? YES

Notes on Pupil Page 123

Tasks: *Pupils complete algebraic sentences.*

Purpose: To teach pupils to interpret algebraic expressions illustrated by diagrams.

Unifying Ideas: Structure; functions and relations; addition and subtraction; multiplication and division; geometry; measurement.

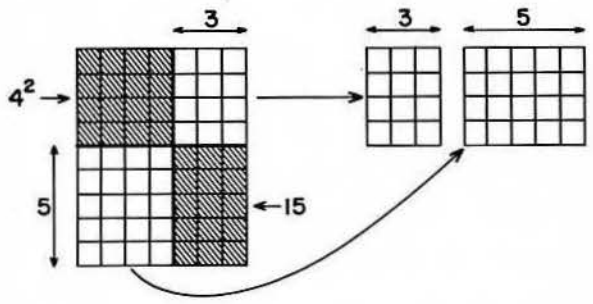
The Lesson: On this page, we replace the dots with regions. But if we write $3x$ in a rectangle, it means we imagine x rows of 3 dots in each. It is also possible to think in terms of area, so that we can say a rectangular region x inches by *3* inches is covered by $3x$ inch-squares.

To make sure that pupils do not confuse area with perimeter, draw a rectangle 3 inches by 5 inches and ask for its area. Some pupils may answer 16 inches—the distance around the edges. If so, ask them to make a number of inch-squares to see how many they would need to paste over the rectangle in order to cover it. It might be helpful if the terms "area" and "pasting over" were to become closely associated in pupils' minds. (The idea of area is essential for the work on pages 126 and 127.)

Pupil page 123 is similar to page 122. Pupils write an expression for the number of dots in (or the area of) a complete region. Such a region is broken into parts. For example, the first rectangle is $x+5$ by $x+3$. Its total area is $(x+5)\ (x+3)$. But the rectangle consists of four parts, with areas x^2, $3x$, $5x$, and 15. Adding these, we have $x^2+8x+15$. This suggests the generalization that, for each x,

$$(x+3)\ (x+5)=x^2+8x+15.$$

If any pupil questions the equivalence of $3x+5x$ and $8x$, you can discuss this in various ways: (1) you can go back to dots and use the same arguments as on page 122; (2) you can choose some number for x, say 4, and demonstrate with graph paper:

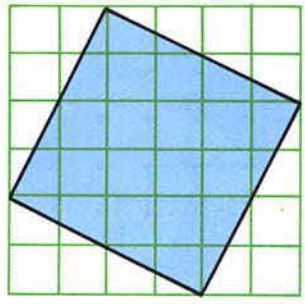


Using pieces of graph paper 3×4 and 5×4, it is possible to make a rectangle 8×4; (3) you can challenge pupils to try to think of a number such that $3x+5x$ is different from $8x$.

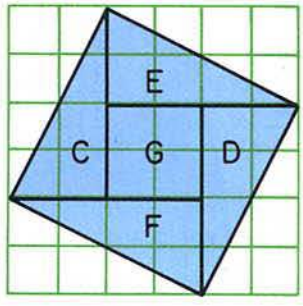
The next two sketches in the left-hand column are simple. A rectangle is divided into two pieces. The sum of the areas of the two parts is written. The area of the whole region is written. This gives us two different ways of writing the same number.

The remaining sketches involve halving an area. These sketches lead to the conclusion that the area of a triangle is half the area of a certain rectangle. Pupils should not merely parrot a formula. For each given triangle, they should sketch the rectangle of which the triangle covers one half. Using common sense, they should be able to find the area of the rectangle. How many inch-squares would be needed to cover it? They should then be able to see that the area of the triangle is just one-half the area of the rectangle.

How many small squares ☐ would you need to cover the large blue square below? (You would need to cut some up into odd shapes.)

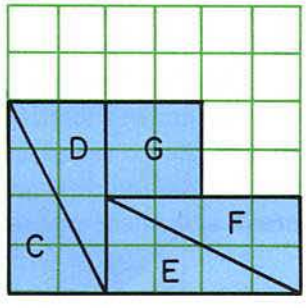


One way to think about the question is to see the square as made up of four triangles and a small square.



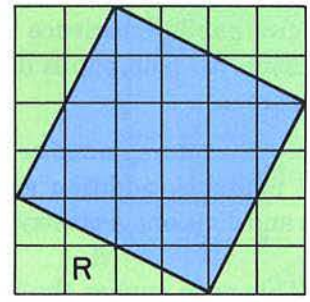
Each blue region is labeled so we can talk about it.

These four triangles and the square can be rearranged as in the following diagram:

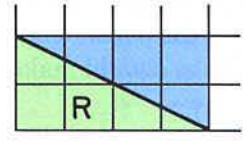


Now you can easily count the number of small squares you would need to cover the blue region.

Here is another approach to the same problem. In the entire diagram, there are 6×6, or 36 small squares. If we can find the number of squares we would need to cover the green regions below, we can get to our result. We would simply subtract the number of whole squares that are green from 36. That will tell us all we need to know about the blue squares.



The four green regions are of the same size. So, let's consider only one.



The blue and green shaded regions exactly cover eight squares. So it must take four square pieces to cover R.

$$4 \times 4 = 16$$

$$36 - 16 = 20$$

So again we have found the answer to our question. We would be surprised if we arrived at different answers.

On the next page you can use either method you like to find the size of the large blue regions in each example.

After you have completed most of the chart at the bottom of the next page, look for an important pattern.

In each column, how can you know the result that will appear in the bottom row from the two results above it?

Notes on Pupil Page 124

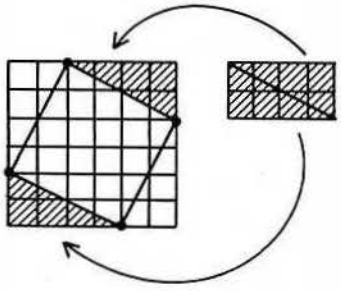
Tasks: *Pupils use two different methods to determine the area of the large blue regions.*

Purpose: To give pupils experience with area measurement. To help pupils discover the Pythagorean Theorem.

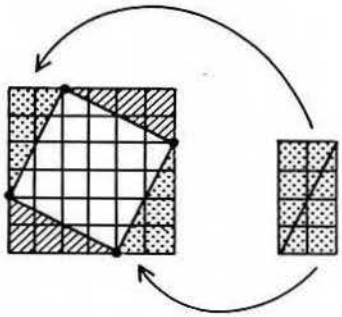
Unifying Ideas: Structure; number and counting; functions and relations; multiplication and division; geometry; measurement.

The Lesson: In the first method we cut the region into pieces and rearrange the pieces for easier measurement. In other words, we use addition; we add the areas of C, D, E, F, and G.

In the second method we use subtraction. The tiled square is drawn on a 6×6 piece of graph paper. So, 36 tiles would be needed to cover it. We have to decide how many tiles would be needed to cover the triangles at the corners. It may help to consider these triangles in pairs. With 8 tiles, I can cover two corners. I have now used 8 of my 36 tiles. I have 28 left.

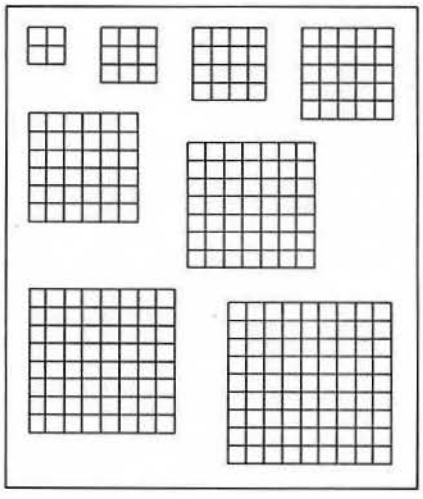


With 8 more tiles, I cover the other two corners. This leaves me with $28 - 8$, or 20, tiles to cover the square in the middle.

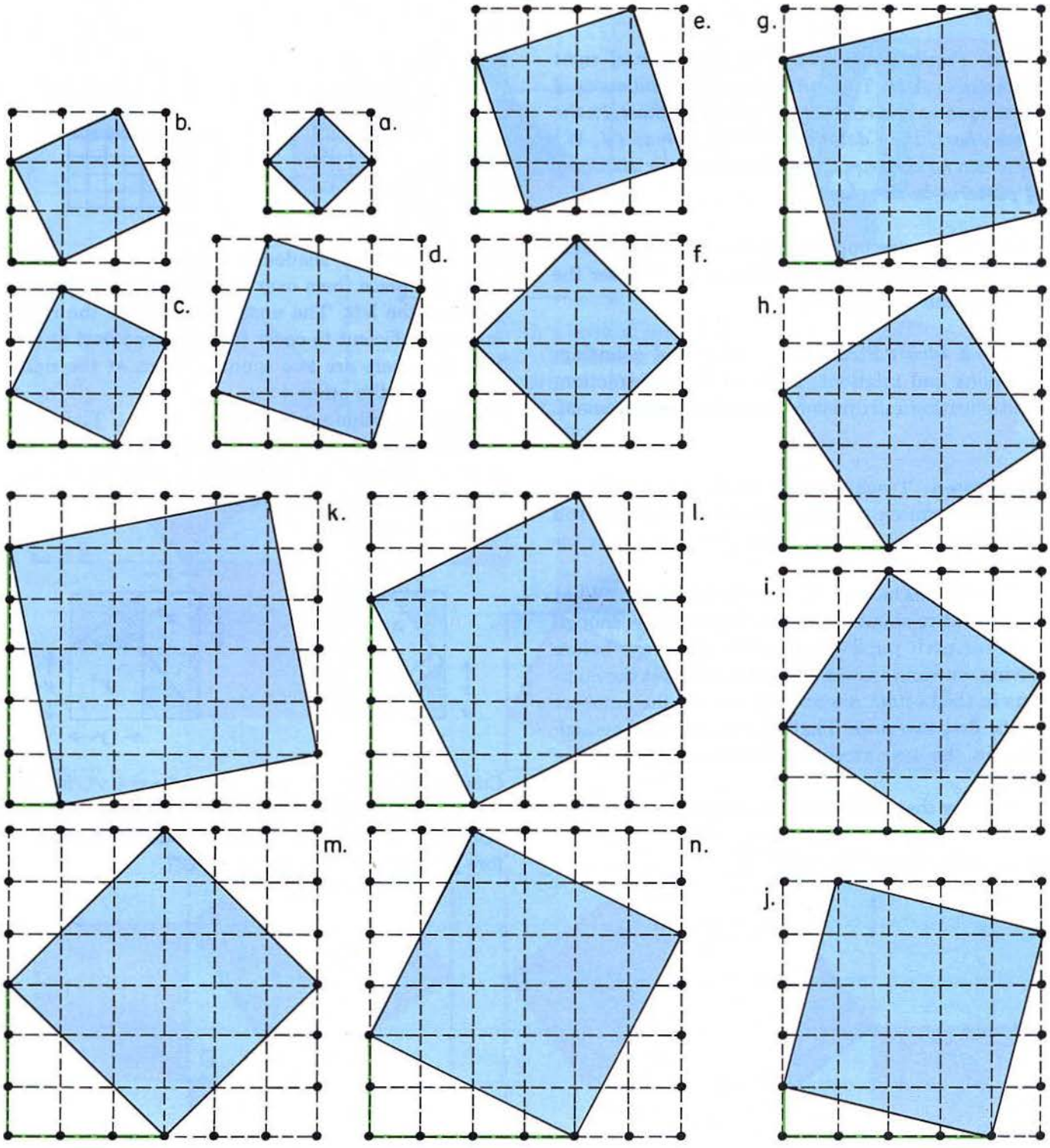


This answer will agree with that found by the first method. Pupils should be familiar with the second method, since it will be useful later on. See, for example, the demonstration described at the end of the Notes on Pupil Page 126.

It will help pupils understand area if you demonstrate both methods using paper or cardboard. We also suggest that you duplicate several square arrays of blocks, preferably on papers of different color to make paste-ups easier to understand. Prepare enough for each pupil to make his own paste-up demonstrations of the ideas discussed here and on pupil pages 125 and 126.



Complete the record below. Indicate the length of green lines in units of •—•. Record the number of small squares required to cover each large blue square.



	a	b	c	d	e	f	g	h	i	j	k	l	m	n	x	y	z
No. of ⁞	1	2	1	1	3	2	4	3	2	1	5	4	3	2	6	7	3
No. of •—•	1	1	2	3	1	2	1	2	3	4	1	2	3	4	8	2	4
Squares shaded blue	2	5	5	10	10	8	17	13	13	17	26	20	18	20	100	53	25

Notes on Pupil Page 125

Tasks: *Pupils find the area of the square region on the hypotenuse of right triangles.*

Purpose: To give pupils experience with linear and area measurement. To help pupils discover the Pythagorean Theorem.

Unifying Ideas: Structure; number and counting; functions and relations; addition and subtraction; multiplication and division; geometry; measurement.

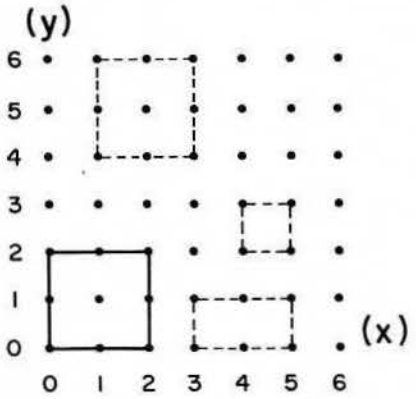
The Lesson: This page applies the ideas presented on pupil page 124. Pupils find the areas of a number of blue squares, using the methods described on pupil page 124 or any methods of their own that will produce correct answers.

Pupils may have to draw cases *x*, *y*, and *z* for themselves on separate squared paper, since there is not enough room on the page. For any exercise that a pupil finds difficult, he should make his own large-scale drawing.

The rule that lies behind the entries in the table will be discovered by pupils on page 126. Do not give it away by telling them what it is!

Each point on a grid can be associated with an ordered pair of numbers. Thus, the bottom square in the grid below can be described by its four corners, or vertices, as: (0, 0), (2, 0), (2, 2), and (0, 2). As enrichment, have pupils use various rules to move the square around the grid, distort it, enlarge it, or shrink it.

Rule A moves the square up and to the right. Rule B slides it to the right and squashes it down into a rectangle. Rule C moves it up and to the right, and squeezes it into a square one-fourth its original size. Had a triangle been used, it could have been turned upside-down, twisted 90° one way or the other, and stretched out.



Rule A:
(x,y)→(x+1,y+4)

x	y	x+1	y+4
0	0	1	4
0	2	1	6
2	2	3	6
2	0	3	4

Rule B:
(x,y)→(x+3,y÷2)

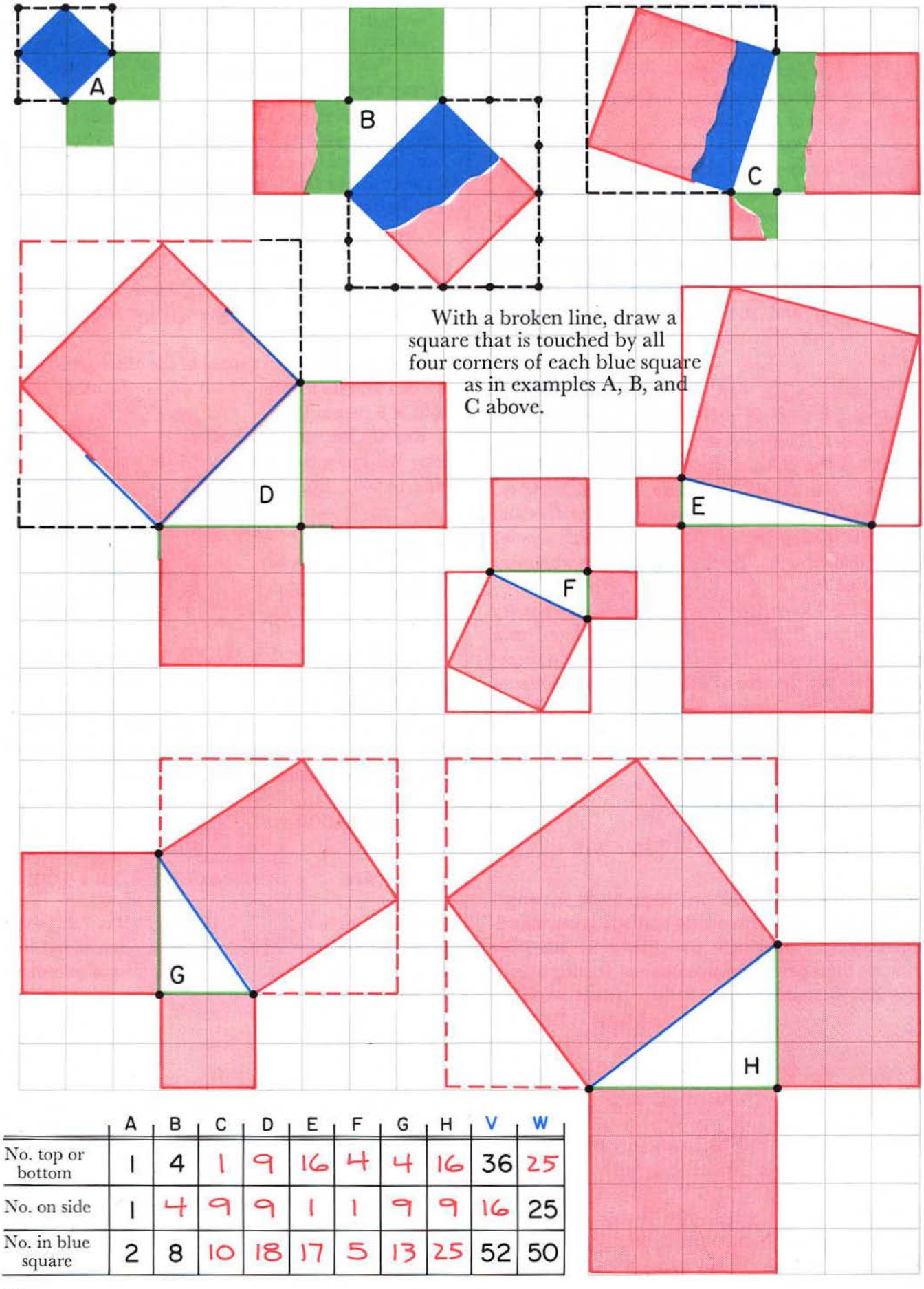
x	y	x+3	y÷2
0	0	3	0
0	2	3	1
2	2	5	1
2	0	5	0

Rule C:
(x,y)→(x÷2+4, y÷2+2)

x	y	x÷2+4	y÷2+2
0	0	4	2
0	2	4	3
2	2	5	3
2	0	5	2

In all of this activity, pupils are learning properties of any shape they use. They become familiar with using ordered pairs of numbers for the points on a grid and begin to feel the power of a formula. In addition, they learn to use—and distinguish between—square units (area) and linear units (perimeter).

Draw blue squares with blue lines as sides. Draw green squares with green lines.



With a broken line, draw a square that is touched by all four corners of each blue square as in examples A, B, and C above.

	A	B	C	D	E	F	G	H	V	W
No. top or bottom	1	4	1	9	16	4	4	16	36	25
No. on side	1	4	9	9	1	1	9	9	16	25
No. in blue square	2	8	10	18	17	5	13	25	52	50

Notes on Pupil Page 126

Tasks: *Pupils draw squares on each side of right triangles in A-H. They then determine the areas of these squares and record the results in a chart. In the same chart, they determine missing areas (V, W) for which no sketches are given, using their discovery of patterns in the chart.*

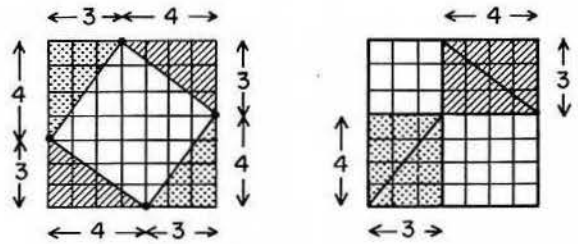
Purpose: To give pupils experiences with perimeter and area measurement. To help pupils discover the Pythagorean Theorem.

Unifying Ideas: Structure; number and counting; functions and relations; addition and subtraction; multiplication and division; geometry; measurement.

The Lesson: Teachers will recognize the Pythagorean Theorem on this page. Pupils presumably will not, but they should be allowed to discover it for themselves.

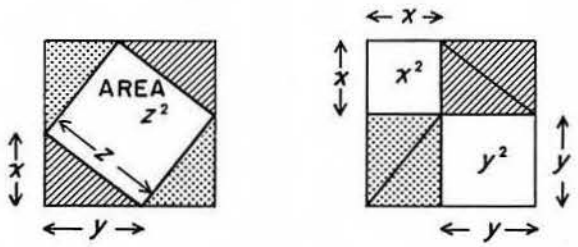
When pupils have completed the table, ask, "What do you notice about the table?" Try to allow enough time for each pupil to make the discovery before having someone answer. The answer is that the numbers in the bottom row are the sums of the numbers in the first two rows. That is, there are just enough tiles in the two green squares to cover the blue square.

To show that a blue square contains as many tiles as its companion two green squares, the following demonstration may be useful.



Cut out the four shaded triangular regions at the right and place them over the shaded triangular regions at the left. The unshaded parts at the right must be sufficient to cover the unshaded part at the left. But there are two square regions at the right —one 3×3, the other 4×4.

There is nothing special about 3 and 4. The same demonstration can be done with *any* numbers. If we write x instead of 3, y instead of 4, and z for the side of the tiled blue square, we are able to state the idea in its general form.



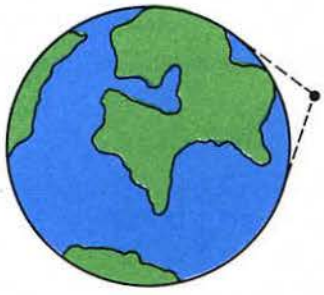
Comparing areas, we note that $z^2 = x^2 + y^2$, for all numbers x, y, and z. This is the Pythagorean Theorem. It is necessary for pupils to understand this before proceeding with pupil page 127.

Before we start, you must realize that we shall, of course, run into some very large distances — and some very large numbers.

Col. John Glenn is the first American astronaut to orbit the earth. Some of the time, he was 100 miles above the earth. How far do you think he could see in all directions?

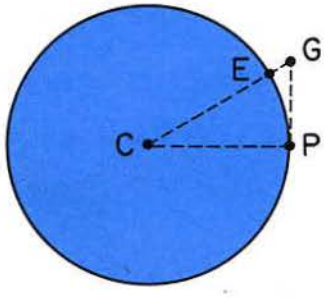
Let's see if we can find out!

Here is a sketch that will help us find out (our sketch is not to scale).



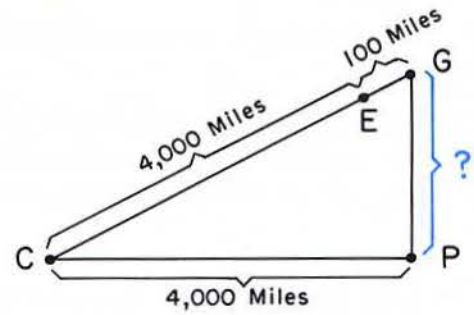
Col. John Glenn's Mercury capsule

If we took a slice right through the center of the earth, we would see the problem this way:



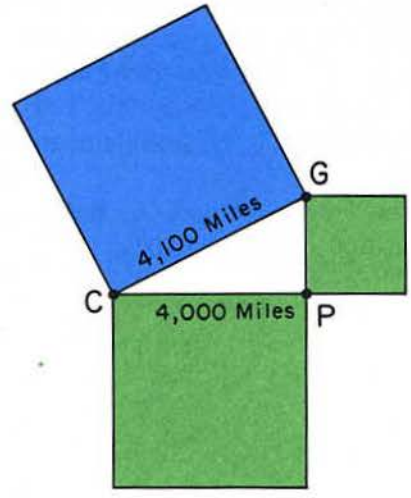
G suggests Glenn, and from point E he is 100 miles above the earth. From that height, he can see as far as point P on the surface of the earth. Of course, point E and point P are about the same distance from C — the center of the earth — and it is about 4,000 miles from any point on the earth to the center of the earth.

Let's draw a larger sketch of that part of our diagram we are most interested in and fill in some distances.



How far is Glenn from the center of the earth? (4,000 + 100 = 4,100.)

He can see from G to P. How far is that?



If the blue square is 4,100 miles on each side, how many square miles are there in the blue square? 16,810,000

How many square miles in the large green square? 16,000,000

How many square miles in the small green square? 810,000

How long is each side of the small green square? 900 MILES

That length is the distance Col. Glenn can see in every direction. (Find out yourself before turning the page.)

Notes on Pupil Page 127

Tasks: *Pupils find the length of one side of a right triangle by finding the area of the square regions on the three sides of the triangle.*

Purpose: To give pupils experiences with linear and area measurement and to show pupils an application of the Pythagorean Theorem.

Unifying Ideas: Structure; number and counting; functions and relations; addition and subtraction; multiplication and division; geometry; measurement.

The Lesson: Pupils may not immediately realize that P is the farthest point on the earth's surface visible from G. Draw a disk on the chalkboard, mark G, and locate a point on the disk opposite G. Could Glenn see this point? No, there is earth between. Now move toward P along the edge of the disk and ask pupils to stop you as soon as you reach a point within Glenn's sight. They should do this when you reach point P.

Angle CPG looks like a right angle. We do not try to *prove* that it is one—that will come later, in a formal geometry. For now, we assume it—and therefore know that the area of the blue square is the sum of the areas of the green squares. So we apply the pattern of the 3, 4, 5 triangle:

$$\text{Green squares} \begin{cases} 3 \times 3 = 9 \\ 4 \times 4 = 16 \end{cases}$$

$$\text{Blue square} - \text{Add} \rightarrow \quad 25 = 5 \times 5$$

Now suppose that we had a problem, in which the blue side is 10 inches long and one green side is 8 inches long. How long is the other green side? We write the same pattern of calculations, leaving blanks for unknown numbers.

$$\text{Green squares} \begin{cases} 8 \times 8 = 64 \\ \quad x \quad = \underline{\quad} \end{cases}$$

$$\text{Blue square} - \text{Add} \rightarrow \quad 100 = 10 \times 10$$

Can we fill in the blanks? Easily: $64 + \underline{\quad} = 100$. The missing area is 36. Now we have:

$$\begin{aligned} 8 \times 8 &= 64 \\ x \quad &= \underline{36} \\ 100 &= 10 \times 10 \end{aligned}$$

Now that we have the square of the other green side, we can find its length: $36 = 6 \times 6$, so the other green side is 6 inches long.

Exactly the same method works for Glenn's problem. We know the blue side (4,100) and one green side (4,000). So, we write:

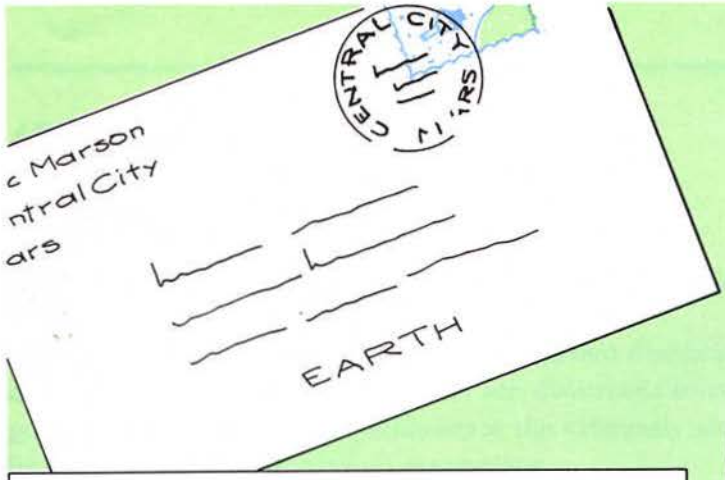
$$\begin{array}{lll} \text{Green} \begin{cases} 4{,}000 \times 4{,}000 = 16{,}000{,}000 \\ \quad x \quad = \underline{\quad\quad\quad\quad} \end{cases} & & \\ \text{Blue} \quad \text{Add} & 16{,}810{,}000 = & \begin{array}{r} 4{,}100 \\ \times 4{,}100 \end{array} \end{array}$$

The missing addend is 810,000, which, fortunately, is exactly 900×900. The answer, therefore, is that P is 900 miles from G.

The same method can be used for any height—but do not expect the arithmetic to be as simple! For example, consider a plane 1 mile up:

$$\begin{aligned} 4{,}000 \times 4{,}000 &= 16{,}000{,}000 \\ x \quad &= \underline{\quad 8{,}001} \\ \text{Add} \quad & 16{,}008{,}001 = 4{,}001 \times 4{,}001 \end{aligned}$$

Now 90×90 is 8,100—too big; 89×89 is 7,921—too small. So the answer becomes "more than 89 but less than 90 miles." This is exact enough for a passenger in an airplane.



Dear Friends,

I'm sorry I had to leave without saying goodbye. On my way home, I worked out the answer to that problem about how far Col. Glenn could see from 100 miles out in space. Here is my computation:

$$\begin{array}{r} 4{,}100 \\ \times\ 4{,}100 \\ \hline 410000 \\ 16400 \\ \hline 16{,}810{,}000 \end{array}$$

$$\begin{array}{r} 4{,}000 \\ \times\ 4{,}000 \\ \hline 16{,}000{,}000 \end{array}$$

$$\begin{array}{r} 16{,}810{,}000 \\ -\ 16{,}000{,}000 \\ \hline 810{,}000 \end{array}$$

$810{,}000 = 900 \times 900$

So, Col. John Glenn could see about 900 miles on Earth in every direction.

This means he could look over about 2,500,000 square miles of Earth scenery.

Your friend,

Alec Marson

P.S. Of course, I could have found the distance Col. Glenn could see by a much easier method that I learned in Algebra. I learned that, for any numbers (I use x and y here, as reminders), it is true that

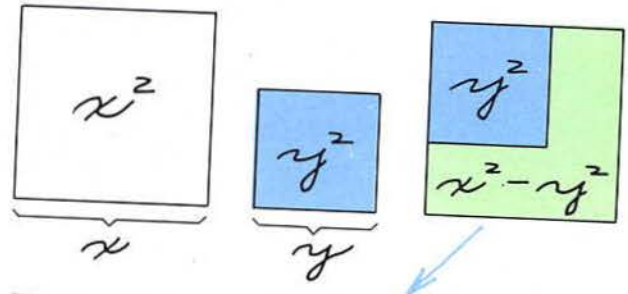
$x^2 - y^2 = (x+y)\cdot(x-y)$

$4{,}100^2 - 4{,}000^2 = (8{,}100)\cdot(100)$

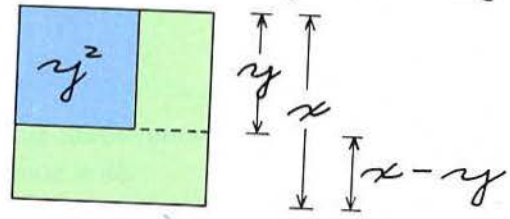
$= 810{,}000$

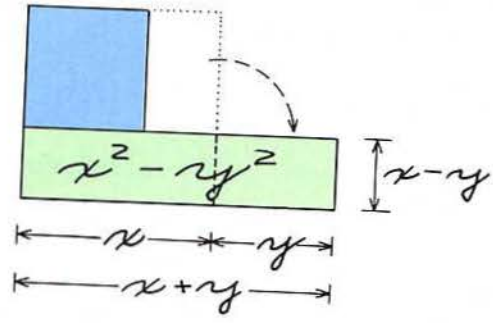
$= 900 \times 900$

Here is a series of sketches I use to help me remember and understand that amazing shortcut.



The green region suggests the idea of $x^2 - y^2$. How big is it?





The green region certainly suggests the idea of $(x+y)\cdot(x-y)$. Clearly $x^2 - y^2 = (x+y)\cdot(x-y)$.

Notes on Pupil Page 128

Tasks: *Pupils study Alec Marson's solution to the problem posed on pupil page 127.*

Purpose: To relate algebraic statements to pictorial representations.

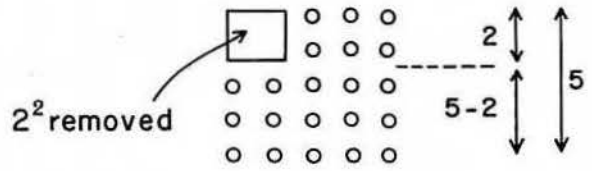
Unifying Ideas: Structure; number and counting; functions and relations; addition and subtraction; multiplication and division; geometry; measurement.

The Lesson: On this page, Alec Marson shows his friends how to solve the problem about Glenn. He begins by subtracting $4{,}000^2$ from $4{,}100^2$ to find 810,000. The reasoning behind these calculations will be understood from the Notes on Pupil Page 127.

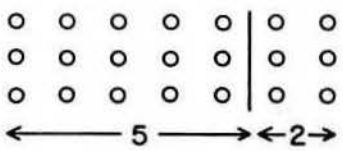
In his postscript, Alec gives a formula for shortening the arithmetic involved: the difference of the squares of two numbers is equal to the sum of the numbers times the difference of the numbers—$x^2-y^2=(x+y)\cdot(x-y)$. Taking 4,100 for x and 4,000 for y, $x+y$ becomes 8,100 and $x-y$ simply 100. So, $(x+y)\cdot(x-y)$ is $8{,}100\times100$, or 810,000. This method saves us the trouble of squaring 4,100 and 4,000.

As in the longer method, we have to finish by seeing that 900×900 is 810,000.

Alec ends by giving some pictures to show that he has not simply memorized the pattern $x^2-y^2=(x+y)\cdot(x-y)$. A pupil should remember that there is a rule for factoring the difference of squares. But, to avoid talking nonsense, he should be able to think out the rule for himself. He may start with a special case, say 5^2-2^2. Here is a picture of 5^2 with 2^2 removed.



What remains can be rearranged as a 3×7 rectangular array:



One side of the array is 7, which is $5+2$. The other side is 3, which is $5-2$. So $5^2-2^2=(5+2)\times(5-2)$.

APPENDIX

THE EXPANDING LANGUAGE OF MATHEMATICS

Many unfamiliar expressions are heard in current discussions of mathematics. Some teachers are interested; others are disturbed; some are confused. The purpose of this appendix is to illustrate the extremely simple ideas indicated by some of the new words and expressions.

Because the concepts dealt with are not at all new, many teachers will prefer to discuss them in the language they have always used. They can do this without any qualms of conscience. All teachers are urged to use new words and expressions *only after* they have a firm comprehension of the ideas, have familiarized themselves with the language, and can use it comfortably. The ideas are important: the language is useful only if it helps present the ideas more clearly.

1. Sets

Mathematicians often borrow words from the general language and give them a specific mathematical meaning. They usually prefer this to inventing words of their own. Such a word is *set*. We are familiar with such expressions as:

"*Set* it on the table."
"The sun will *set* at 5:30 P.M."
"That stage *set* is ingenious."
"I'm going to *set* my hair."
"Let's play a *set* of tennis."
"The cement will *set* by morning."

None of these has much, if anything, to do with the meaning mathematicians have given the word *set*. Their meaning has more in common with:

"How many pieces in the *set* of dishes?"
"Breakfast *sets* for $24.99."
"He has a *set* of ivory chessmen."
"I got a gold-plated pen and pencil *set*."

They looked at the word *set* as it might be used in place of *group*, *collection*, *combination*, *aggregate*, *family*, etc., and then proceeded to give that general idea a precise mathematical meaning.

Sets in Mathematics. Many uses of the word *set* in mathematics have a rather familiar ring. Here are a few in which the word might be replaced by *group* or *collection* as those words are used in everyday discourse.

The set of all counting numbers between 10 and 15 has 11, 12, 13, and 14 as its members.

Mathematicians use braces to suggest a set, so they might indicate this particular set of numbers by:

$$\{11, 12, 13, 14\}$$

A mathematician might wish to direct your attention to all the counting numbers and zero that are multiples of 7, in which case he would write:

$$\{0, 7, 14, 21, 28, \ldots\}$$

The three dots are a notation for "and so on in a like manner."

The members of a set need not be numbers. We might wish to consider, for example:

The set of all cities in the United States with populations in excess of one million.

The set of all objects in my purse.

The set of redheaded boys.

The set of all triangles.

The set of all letters in the alphabet that are never used as vowels.

2. Union of Sets

One way of combining sets is called "union," for which the symbol $\cup$ is most commonly used.

(1) $\{A, B\} \cup \{A, C, E\} = \{A, B, C, E\}$
What letters occur either in the set with members A and B or in the set with members A, C, and E? Answer: the letters A, B, C, and E.

(2) $\{1, 5, 4\} \cup \{4, 6, 1\} = \{1, 4, 5, 6\}$ (or $\{4, 6, 5, 1\}$, etc.)

(3) $\{Al, Ed\} \cup \{Carl, Ed\} = \{Al, Ed, Carl\}$

(4) $\{m, n\} \cup \{r, s, t\} = \{m, n, r, s, t\}$

(5) $\{7, 2, 9\} \cup \{9, 2\} = \{7, 2, 9\}$

(6) $\{a, b, c\} \cup \{c, b, a\} = \{a, b, c\}$

(7) $\{b, x, y\} \cup \{x\} = \{\_\_, \_\_, \_\_\}$

(8) $\{Jane\} \cup \{Jane, Mary\} = \{\_\_, \_\_\}$

(9) $\{P, Q\} \cup \{R, S, T\} = \{\_\_, \_\_, \_\_, \_\_, \_\_\}$

(10) $\{(3, 4), (7, 9)\} \cup \{(7, 9), (4, 3)\} = \{(3, 4), (7, 9), (4, 3)\}$

(11) $\{(7, 0)\} \cup \{(0, 7)\} = \{(7, 0), (0, 7)\}$

(12) $\{(7, 0)\} \cup \{(7, 0)\} = \{(7, 0)\}$

(13) $\{(1, 8)\} \cup \{(8, 1)\} = \{(\ ,\ ), (\ ,\ )\}$

(14) $\{(0, 0), (0, 1), (1, 1)\} \cup \{(1, 0), (0, 1)\} = \{(\ ,\ ), (\ ,\ ), (\ ,\ ), (\ ,\ )\}$

(15) $\{(a, b), (b, a)\} \cup \{(b, b), (a, b)\} = \{\_\_\_\_\_\}$

(16) {r, s, t} ∪ {s, t, a, r} = {_____}
(17) {17, 23, 47} ∪ {32, 47, 71} = {____, ____, ____, ____, ____}
(18) {(1, 1), (2, 2)} ∪ {(2, 1), (1, 1)} = {_____}

3. Intersection of Sets

Another way of combining sets is called "intersection," for which the symbol ∩ is most commonly used.

(1) {R, S, T} ∩ {T, O} = {T}
What letters occur both in the set with members R, S, and T and in the set with members T and O? Answer: Only the letter T.
(2) {a, b, c, d} ∩ {c, a, e} = {a, c}
(3) {5, 7, 3} ∩ {57, 73, 7} = {7}
(4) {a, b, d} ∩ {b, e, a, d} = {a, b, d} (or {b, a, d}, etc.)
(5) {3, 6, 9, 12, 18} ∩ {4, 8, 12, 16} = {12}
(6) {(0, 0), (1, 5), (2, 10)} ∩ {(5, 1), (1, 5)} = {(1, 5)}
(7) {Bill, Sam, Tom} ∩ {Tom, Sam, Ed} = {Sam, Tom}
(8) {Bill, Dick, Tom} ∩ {Carl, Dick} = {_____}
(9) {(7, 1), (5, 3), (0, 8)} ∩ {(1, 7), (0, 8)} = {(,)}
(10) {8, 4, 9, 1} ∩ {4, 1, 9, 8} = {_____}
(11) {(28, 3), (23, 8)} ∩ {(23, 8), (28, 3)} = {_____}
(12) {(0, 1), (1, 0), (1, 2)} ∩ {(2, 1), (1, 0), (0, 1)} = {_____}
(13) {0, 2, 4, 6, 8, 10, 12} ∩ {0, 3, 6, 9, 12, 15} = {_____}
(14) {1, 3, 5, 7, 9, 11} ∩ {2, 4, 6, 8, 10, 12} = {_____}
In example 14 there are no members common to both sets, so the intersection in this case is called "the empty set."

4. The Empty Set

While this expression sounds a bit strange at first, the idea is very simple, as you can see by referring to example 14 above. The symbol used to indicate the empty set is ∅. So we might say: the empty set = { } = ∅.

(1) (A, R, T) ∩ (B, C) = { } = ∅
Which letters occur both in the set with members A, R, and T and in the set with members B and C? Answer: none. So we say that the intersection is the empty set.
(2) {5, 7} ∩ {0, 6, 12} = ∅
(3) {a, b} ∪ {c} = {a, b, c}
(4) {a, b} ∩ {c} = ∅
(5) {3, 7, 11} ∩ { } = ∅
Which numbers are both in the set with members 3, 7, and 11 and the empty set? Answer: none. Since the symbol { }—braces with nothing listed between them—might be confusing, the empty

set is usually indicated by the symbol $\varnothing$. So $\{\quad\} \cap \{\quad\} = \{\quad\}$ would be written: $\varnothing \cap \varnothing = \varnothing$.

(6) $\varnothing \cap \{1, 2, 3\} =$

(7) $\varnothing \cup \{1, 2, 3\} = \{1, 2, 3\}$

(8) $\{(0, 8), (9, 1)\} \cap \{(9, 8)\} =$

(9) $\{\text{Alice, Mary}\} \cap \{\text{Bill}\} =$

(10) $\{4, 8, 12\} \cup \{0, 5, 10\} = \{\_\_, \_\_, \_\_, \_\_, \_\_, \_\_\}$

(11) $\{4, 8, 12\} \cap \{0, 5, 10\} =$

5. Number of a Set

We are often concerned with the number of members in a set. Since we are referring to numbers, we can add and subtract. We write a capital N before the name of the set to indicate that we are concerned only with the number of members in the set.

(1) $N\ \{1, 4, 17\} = 3$

(2) $N\ \{a, b, x, y, z\} = 5$

(3) $N\ \{a, b, c\} + N\ \{x, y, z\} = 3 + 3 = 6$

(4) $N\ \{a, b, c\} + N\ \{a, b, c\} = 6$

(5) $\{a, b, c\} \cup \{a, b, c\} = \{a, b, c\}$

(6) $N\ \{4, 6, 8, 9\} + N\ \{6, 8, 11\} = 7$

(7) $\{4, 6, 8, 9\} \cup \{6, 8, 11\} = \{4, 6, 8, 9, 11\}$

(8) $\{4, 6, 8, 9\} \cap \{6, 8, 11\} = \{6, 8\}$

(9) $N\ \{x, y, z, a\} - N\ \{1, 2, 4\} = 1$

(10) $N\ \{0, 1, 2, 3\} + N\ \{0\} = 5$

(11) $N\ \{\quad\} = N\ \varnothing = 0$

6. Solution Sets for Mathematical Sentences

In the following examples let us agree to limit ourselves to the counting numbers and zero. Thus, for the open sentence:

$\triangle + \square = 2$

there are only three pairs of numbers that could be used to convert it into a true sentence:

$(0, 2) \ldots\ldots 0 + 2 = 2$
$(1, 1) \ldots\ldots 1 + 1 = 2$
$(2, 0) \ldots\ldots 2 + 0 = 2$

The *solution set* of this open sentence is the set whose members are (0, 2), (1, 1), and (2, 0).

In this case, where pairs of numbers are used, we must agree that the first member of the pair is associated with the first placeholder in the mathematical sentence, and that the second member is associated with the second placeholder. Such a pair is called an *ordered pair*. Since addition is commu-

tative, this seems rather trivial, but for the operations that are not commutative this idea becomes very important.

Here are additional examples of solution sets.

Examples	Open Sentence	Solution Set
(1)	$\triangle + \square = 2$	$\{(0,2),(1,1),(2,0)\}$
(2)	$\square \times 1 = \square$	$\{0, 1, 2, 3, 4, \ldots\}$

because $0 \times 1 = 0$ $\quad 3 \times 1 = 3$

$1 \times 1 = 1$ $\quad 4 \times 1 = 4$

$2 \times 1 = 2$ $\quad$ etc.

(3)	$\triangle - \triangle = 0$	$\{0, 1, 2, 3, 4, \ldots\}$
(4)	$\square + \square = \square \times \square$	$\{0, 2\}$

because $0+0 = 0 \times 0$

$2+2 = 2 \times 2$

There are just two numbers in the solution set for example 4 because 0 and 2 are the only numbers whose doubles are the same as their squares.

(5)	$\square - \triangle = 6$	$\{(6,0),(7,1),(8,2),\ldots\}$

because $6 - 0 = 6$

$7 - 1 = 6$

$8 - 2 = 6$ $\quad$ etc.

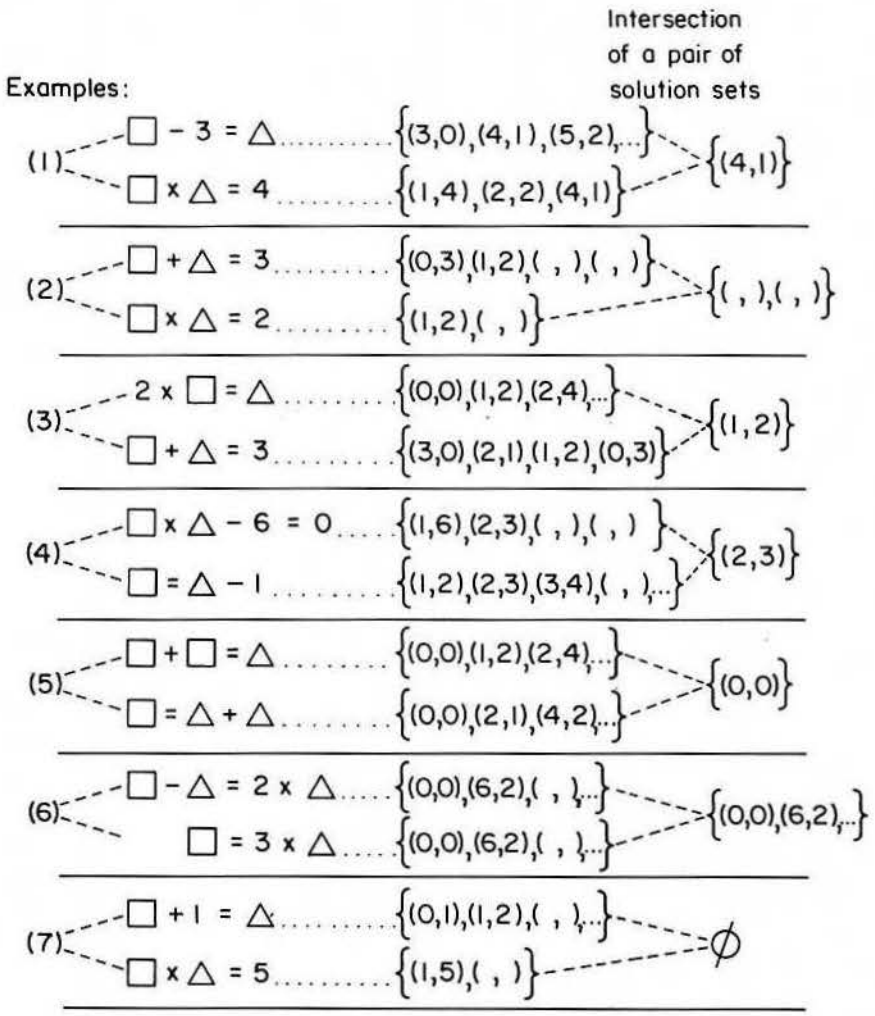
(6)	$\square - 1 = \square$	$\emptyset$

In example 6, the solution set is the empty set because there is no number which is 1 less than itself:

$0 - 1 = 0$ is a false statement
$1 - 1 = 1$ is a false statement
$2 - 1 = 2$ is a false statement
$3 - 1 = 3$ is a false statement
etc.

(7)	$\square + \triangle = \triangle + \square$	All pairs of numbers
(8)	$\square + 9 = 24$	$\{15\}$
(9)	$10 - \triangle = 7$	$\{\quad\}$
(10)	$3 \times \square - 1 = 14$	$\{\quad\}$
(11)	$\triangle \times \square = 6$	$\{(\ ,\ ),(\ ,\ ),(\ ,\ ),(\ ,\ )\}$

7. Intersection of Solution Sets



8. Subsets of Sets

Examples:

Consider Set A = {M, N}. There are four subsets of Set A. They are {M}, {N}, {M, N}, and ∅.

Consider Set B = {Carl}. There are two subsets of Set B. They are {Carl} and ∅.

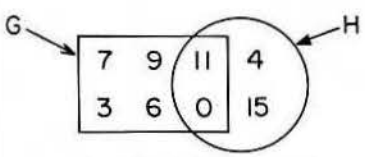
Consider Set C = {1, 2, 3}. There are eight subsets of Set C. They are {1}, {2}, {3}, {1, 2} {1, 3}, {2, 3}, {1, 2, 3}, and ∅.

Were you tempted to say that {N, M} is a subset of Set A? Or that {3, 2, 1} is a subset of Set C? Here order does not concern us, for {M, N} and {N, M} are the same set.

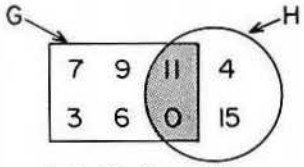
Notice that (a) every set is a subset of itself
and
(b) the empty set is a subset of every set.

9. AND and OR

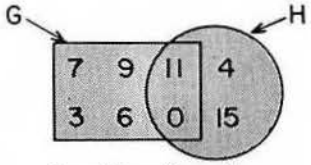
Many of our examples depended on the distinction between the meanings of *and* and *or* as mathematicians use these words. We can explore these ideas in diagrams. Suppose that we are talking about G, the set of numbers written in the rectangle, and H, the set of numbers written in the circle.



G = (7,9,11,3,6,0)
H = (11,4,0,15)

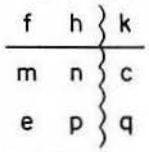


in both G and H
G ∩ H = (11, 0)

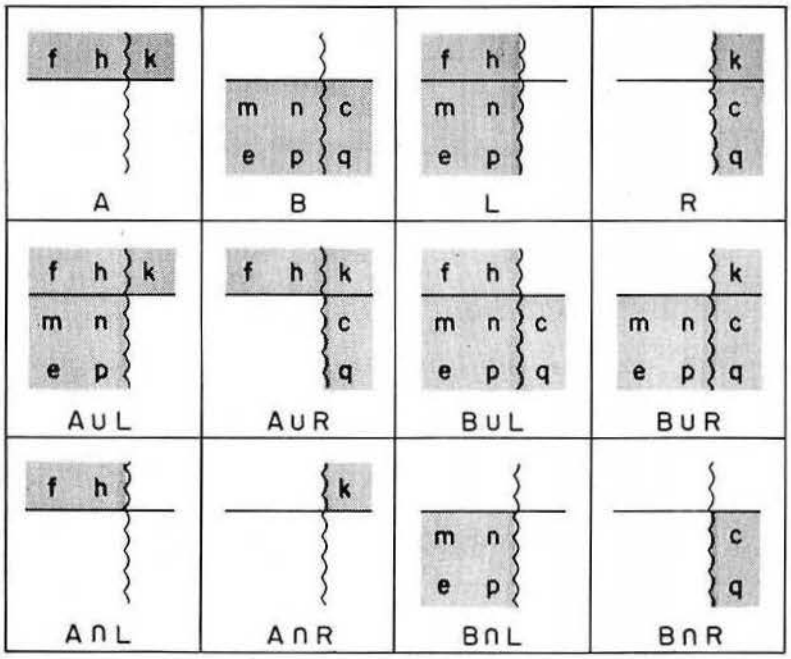


in either G or H
G ∪ H = (7,9,11,4,3,6,0,15)

Let us consider another diagram, involving nine letters.



A = the set of letters above the horizontal line = {f, h, k}
B = the set of letters below the horizontal line = {m, n, c, e, p, q,}
L = the set of letters to the left of the wavy line = {f, h, m, n, e, p}
R = the set of letters to the right of the wavy line = {k, c, q}



$A \cup L = \{f, h, k, m, n, e, p\}$
$A \cap L = \{f, h\}$
$A \cup R = \{f, h, k, c, q\}$
$A \cap R = \{k\}$
$B \cup L = \{\_\_\_\_\_\_\_\_\_\}$
$B \cap L = \{\_\_\_\_\_\_\_\}$
$B \cup R = \{\_\_\_\_\_\_\_\_\_\_\}$
$B \cap R = \{\_\_\_\_\}$
$A \cup B = \{f, h, k, m, n, c, e, p, q\}$
$A \cap B = \varnothing$
$L \cup R = \{\_\_\_\_\_\_\_\_\_\_\_\_\_\}$
$L \cap R =$

This interpretation of *and* and *or* is a mathematical interpretation, but in many ways this usage conforms with conversational English. The idea is disarmingly simple. Once children grasp the concepts indicated by these two words, we have a way to ask many questions about a single example such as the diagram above.

10. AND—OR Games

Let us define *and* and *or* with a series of examples. How many of the following words contain the letter A? How many contain the letter T? How many contain the letter A *and* the letter T? How many contain the letter A *or* the letter T?

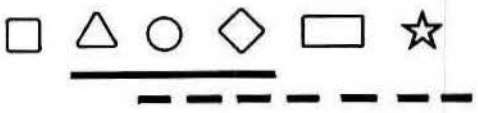
MAN	ATE
HIM	THE
CUT	ASK
BAT	BET

We shall list the groups we are interested in:

A: MAN, BAT, ATE, ASK (4)
T: CUT, BAT, ATE, THE, BET (5)
A *and* T: BAT, ATE (2)
A *or* T: MAN, CUT, BAT, ATE, ASK, THE, BET (7)

The group that contains A *and* T contains all the words that occur in *both* of the first two lists. The group that contains A *or* T contains all the words that occur at least once in *either* of the first two lists.

Another example. How many shapes below are underlined with the solid line? How many are underlined with the broken line? How many are underlined with the solid line *and* the broken line? How many are underlined with the solid line *or* the broken line?



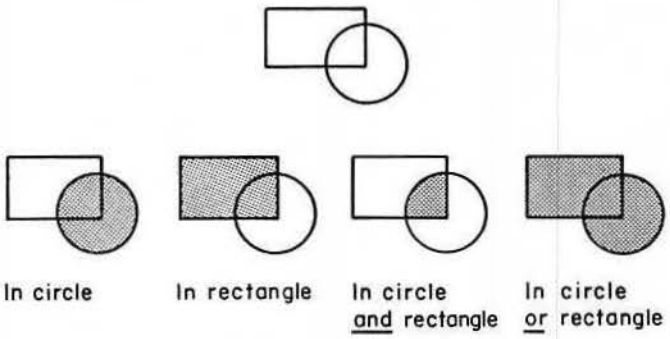
Solid: △ ○ ◇ (3)

Broken: ○ ◇ ▭ ☆ (4)

Solid and broken: ○ ◇ (2)

Solid or broken: △ ○ ◇ ▭ ☆ (5)

Another example. Shade the area that is in the circle; in the rectangle; in the circle *and* in the rectangle; in the circle *or* in the rectangle.



Another example. List all whole-number multiples of 3 (including 0) that are less than 25. Call it List D. List all whole-number multiples of 2 (including 0) that are less than 25. Call it List F.

List D: 0, 3, 6, 9, 12, 15, 18, 21, 24
List F: 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 24

Now, list all the numbers that are in List D *and* List F:

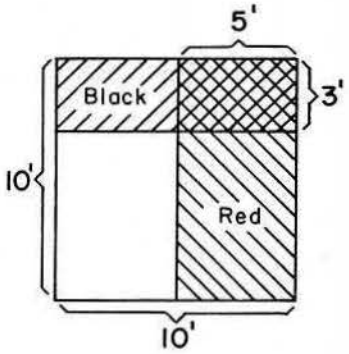
0, 6, 12, 18, 24

List all the numbers that are in List D *or* List F:

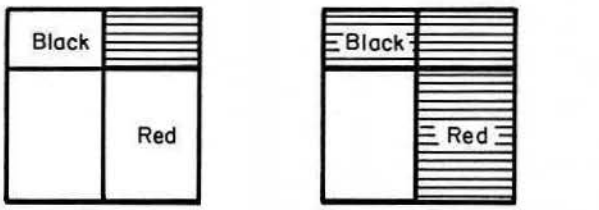
0, 2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24

Another example. List all the odd numbers less than 100. List all the whole number multiples of 7 (including 0) that are less than 100. How many numbers are in the first list? (Fifty) How many numbers are in the second list? (Fifteen) How many numbers are in the first list *and* the second list? (Seven) How many are in the first list *or* the second list? (Fifty-eight)

Here is another example to help define *and* and *or* as these two words are used in mathematics. Below is a sketch of a floor that is 10 feet by 10 feet. A strip 3 feet wide was painted black along one side. Then a strip 5 feet wide was painted red along an adjacent side.



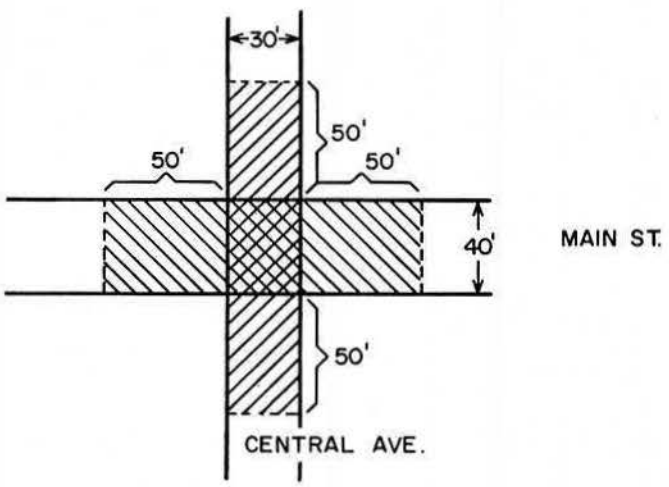
What is the area of the floor? (100 square feet)
What is the area that was painted black? (30 square feet)
What is the area that was painted red? (50 square feet)
What is the area that was painted black *and* red? (15 square feet)
What is the area that was painted black *or* red? (65 square feet)



The word *and* suggests such ideas as "overlap," "both at once"—the *intersection* of areas or groups or lists.

The word *or* suggests such ideas as "in all," "either one or both"—the *union* of areas or groups or lists.

This language suggests another example. Ace Contractors were awarded a contract to replace the paving where Main Street crosses Central Avenue — to a distance of 50 feet from the corner in all four directions. Main Street was 40 feet wide and Central Avenue was 30 feet wide.



How much paving was laid on Main Street? (40×130, or 5,200, square feet)

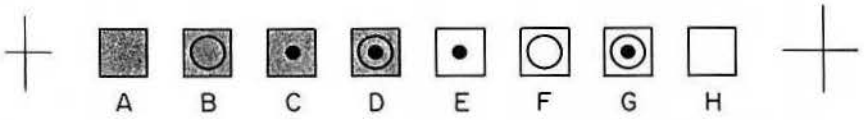
How much was repaved on Central Avenue? (30×140, or 4,200, square feet)

How much paving was replaced in the "intersection"? (30×40, or 1,200, square feet)

How much paving was replaced in all—how much paving is in the "union" of all the area covered by the contract? (8,200 square feet)

Why is the "union" or the total area not 5,200+4,200, or 9,400, square feet? (Because they paved the "intersection" only once.)

Here is another example. Below are some blue blocks and some green blocks:



How many blue blocks have dots? Obviously, the design labeled C is one that fits the conditions of the question. How about D? We are going to accept it also, because it would answer "yes" when asked, "Do you contain a dot?" even though it also contains a circle.

A further question: How many designs are without a dot *or* without a circle? Suppose that each design is asked a pair of questions:

1. Are you without a dot?
2. Are you without a circle?

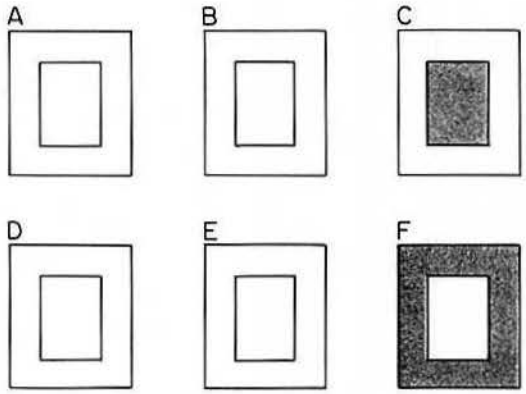
In answer to question 1, how many could answer "yes"? (A, B, F, H)

In answer to question 2, how many could answer "yes"? (A,C, E, H)

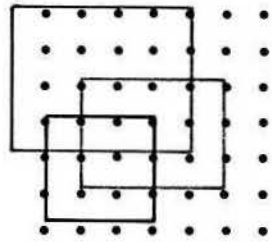
How many could answer "yes" to either one *or* the other (or both)? (A, B, C, E, F, H—six) So, six designs are without dots *or* without circles.

Had the original question been: "How many designs are without dots *and* without circles?" then we would find out how many could answer "yes" to *both* questions 1 and 2. Only two designs (A, H) could answer "yes" to both questions.

Prepare a set of playing cards that look like this:



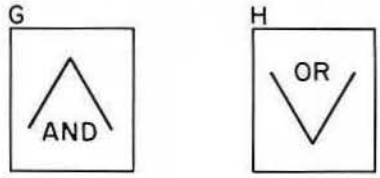
to be used with this playing board (which can be varied):



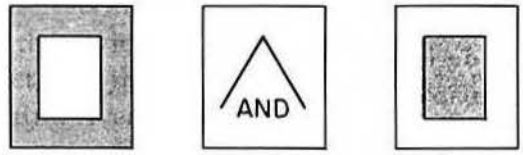
Since the interior of card A is shaded red, it represents the dots enclosed by (*in*) the red on the playing board. Since the exterior of card D is shaded red, it represents the dots *not in* the red on the playing board.

Card A: the number of dots *in* red ($4 \times 5 = 20$)
Card B: the number of dots *in* green ($3 \times 4 = 12$)
Card C: the number of dots *in* blue ($3 \times 3 = 9$)
Card D: the number of dots *not in* red ($49 - 20 = 29$)
Card E: the number of dots *not in* green ($49 - 12 = 37$)
Card F: the number of dots *not in* blue ($49 - 9 = 40$)

Two other cards (preferably several of each) are required:

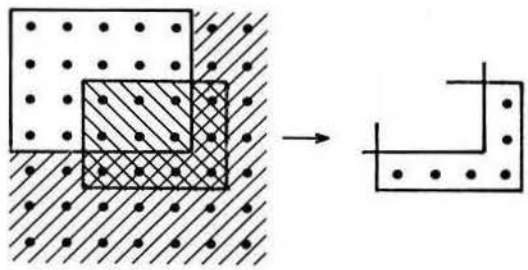


A player selects two colored cards and one of the *and/or* cards. He then places one colored card on each side of the *and* or the *or* card. It might look like this:



Now consider the playing board. How many dots could answer "yes" to both of the following questions?

1. Is it true that you are *not in* red?
2. Is it true that you are *in* green?



Only six dots could answer "yes" to both of these questions—and that is the player's score: 6.

Had the center card been *or* instead of *and*, the question would be: How many dots could answer "yes" to either question 1 or question 2? Then the player's score would be 35 (49—14 = 35)—all the dots in any part of the region shaded in the sketch above.

Players take turns and each accumulates a score. The first player to reach a predetermined total score wins the game. There are seventy-two possible combinations that can be summarized as follows:

∨	R	G	B	R̸	G̸	B̸
R	20	26	26	49	42	43
G	26	12	17	35	49	44
B	26	17	9	32	41	49
R̸	49	35	32	29	43	46
G̸	42	49	41	43	37	45
B̸	43	44	49	46	45	40

∧	R	G	B	R̸	G̸	B̸
R	20	6	3	0	14	17
G	6	12	4	6	0	8
B	3	4	9	6	5	0
R̸	0	6	6	29	23	23
G̸	14	0	5	23	37	32
B̸	17	8	0	23	32	40

R stands for IN red ; R̸ stands for NOT IN red.
etc.

Pupils ought to play the game without these two charts so that each play must be thought out. After a very short time, the ideas of *and* and *or* will become clear.